

η -series and a Boolean Bercovici–Pata bijection for bounded k -tuples

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Abstract

Let $\mathcal{D}_c(k)$ be the space of (non-commutative) distributions of k -tuples of selfadjoint elements in a C^* -probability space. On $\mathcal{D}_c(k)$ one has an operation \boxplus of free additive convolution, and one can consider the subspace $\mathcal{D}_c^{\text{inf-div}}(k)$ of distributions which are infinitely divisible with respect to this operation. The linearizing transform for \boxplus is the R -transform (one has $R_{\mu \boxplus \nu} = R_\mu + R_\nu$, $\forall \mu, \nu \in \mathcal{D}_c(k)$). We prove that the set of R -transforms $\{R_\mu \mid \mu \in \mathcal{D}_c^{\text{inf-div}}(k)\}$ can also be described as $\{\eta_\mu \mid \mu \in \mathcal{D}_c(k)\}$, where for $\mu \in \mathcal{D}_c(k)$ we denote $\eta_\mu = M_\mu/(1 + M_\mu)$, with M_μ the moment series of μ . (The series η_μ is the counterpart of R_μ in the theory of Boolean convolution.) As a consequence, one can define a bijection $\mathbb{B} : \mathcal{D}_c(k) \rightarrow \mathcal{D}_c^{\text{inf-div}}(k)$ via the formula

$$R_{\mathbb{B}(\mu)} = \eta_\mu, \quad \forall \mu \in \mathcal{D}_c(k). \quad (\text{I})$$

We show that \mathbb{B} is a multi-variable analogue of a bijection studied by Bercovici and Pata for $k = 1$, and we prove a theorem about convergence in moments which parallels the Bercovici-Pata result. On the other hand we prove the formula

$$\mathbb{B}(\mu \boxtimes \nu) = \mathbb{B}(\mu) \boxtimes \mathbb{B}(\nu), \quad (\text{II})$$

with μ, ν considered in a space $\mathcal{D}_{\text{alg}}(k) \supseteq \mathcal{D}_c(k)$ where the operation of free multiplicative convolution \boxtimes always makes sense. An equivalent reformulation of (II) is that

$$\eta_{\mu \boxtimes \nu} = \eta_\mu \boxstar \eta_\nu, \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k), \quad (\text{III})$$

where \boxstar is an operation on series previously studied by Nica and Speicher, and which describes the multiplication of free k -tuples in terms of their R -transforms. Formula (III) shows that, in a certain sense, η -series behave in the same way as R -transforms in connection to the operation of multiplication of free k -tuples of non-commutative random variables.

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1. Introduction

The extent to which developments in free probability parallel phenomena from classical probability has exceeded by far what was originally expected in this direction of research. In particular there exists a well-developed theory of infinitely divisible distributions in the free sense; a few of the papers building this theory are [12], [3], [4] – see also the section 2.11 of the survey [14] for more details. The Boolean Bercovici–Pata bijection is one of the results in this theory (cf [4], Section 6); it is a special bijection between the set of probability distributions on \mathbb{R} which are infinitely divisible with respect to free additive convolution (on one hand), and the set of all probability measures on \mathbb{R} (on the other hand).

In this paper we extend the Boolean Bercovici–Pata bijection to the multi-variable framework, in a context where we deal with k -tuples of bounded random variables (or in other words, we deal with non-commutative multi-variable analogues for distributions with compact support). The framework we consider is thus:

$$\left(\mathcal{D}_c(k), \boxplus \right),$$

where k is a positive integer, $\mathcal{D}_c(k)$ is the set of linear functionals $\mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C}$ which appear as joint distribution for a k -tuple of selfadjoint elements in a C^* -probability space, and \boxplus is the operation of free additive convolution. This operation is considered in connection with the notion of free independence, and it encodes the distribution of the sum of two freely independent k -tuples, in terms of the distributions of the two k -tuples. A more detailed review of $(\mathcal{D}_c(k), \boxplus)$ and of the notations we are using in connection to it appears in the Section 4 of the paper. For a general introduction to the ideas of free probability we refer to [13].

A distribution $\mu \in \mathcal{D}_c(k)$ is said to be infinitely divisible with respect to \boxplus if for every $N \geq 1$ there exists a distribution $\mu_N \in \mathcal{D}_c(k)$ such that the N -fold \boxplus -convolution $\mu_N \boxplus \mu_N \boxplus \dots \boxplus \mu_N$ is equal to μ . The set of distributions $\mu \in \mathcal{D}_c(k)$ which have this property will be denoted by $\mathcal{D}_c^{\text{inf-div}}(k)$.

The linearizing transform for \boxplus is the *R-transform*. The R -transform of a distribution $\mu \in \mathcal{D}_c(k)$ is a power series $R_\mu \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$, where $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ denotes the set of power series with complex coefficients, and with vanishing constant term, in k non-commuting indeterminates z_1, \dots, z_k . The above mentioned linearization property is that

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu, \quad \forall \mu, \nu \in \mathcal{D}_c(k). \quad (1.1)$$

We denote by $\mathcal{R}_c(k)$ the set of power series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ which appear as R_μ for some $\mu \in \mathcal{D}_c(k)$; and, similarly, we use the notation $\mathcal{R}_c^{\text{inf-div}}(k)$ for the set of series which appear as R_μ for a distribution $\mu \in \mathcal{D}_c^{\text{inf-div}}(k)$. A distribution μ is always uniquely determined by its R -transform, so we are dealing in fact with two bijections,

$$\mathcal{D}_c(k) \ni \mu \mapsto R_\mu \in \mathcal{R}_c(k), \quad \mathcal{D}_c^{\text{inf-div}}(k) \ni \mu \mapsto R_\mu \in \mathcal{R}_c^{\text{inf-div}}(k). \quad (1.2)$$

In this paper we put into evidence a commutative diagram where the vertical arrows are the two bijections from (1.2), and where the top horizontal arrow \mathbb{B} is a multi-variable counterpart of the Boolean Bercovici–Pata bijection:

$$\begin{array}{ccc}
\mathcal{D}_c(k) & \xrightarrow{\mathbb{B}} & \mathcal{D}_c^{\text{inf-div}}(k) \\
R \downarrow & \searrow \eta & \downarrow R \\
\mathcal{R}_c(k) & \xrightarrow{\text{Reta}} & \mathcal{R}_c^{\text{inf-div}}(k)
\end{array} \tag{1.3}$$

We observe the somewhat surprising occurrence in this diagram of another kind of transform, the η -series. For $\mu \in \mathcal{D}_c(k)$, the η -series associated to μ is $\eta_\mu = M_\mu/(1 + M_\mu) \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$, where M_μ denotes the moment series of μ . The η -series is the linearizing transform for another kind of convolution on $\mathcal{D}_c(k)$, the *Boolean convolution* \uplus . The operation \uplus is the counterpart of \boxplus in connection to Boolean independence – it encodes the distribution of the sum of two Boolean independent k -tuples, in terms of the distributions of the two k -tuples. The counterpart of Equation (1.1) in the framework of \uplus says that

$$\eta_{\mu \uplus \nu} = \eta_\mu + \eta_\nu, \quad \forall \mu, \nu \in \mathcal{D}_c(k). \tag{1.4}$$

A distribution μ is uniquely determined by η_μ , so if we denote $\mathcal{E}_c(k) := \{f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \mid \exists \mu \in \mathcal{D}_c(k) \text{ such that } \eta_\mu = f\}$, then we have a bijection

$$\mathcal{D}_c(k) \ni \mu \mapsto \eta_\mu \in \mathcal{E}_c(k). \tag{1.5}$$

Note that we have drawn this bijection along the diagonal of the diagram (1.3); the justification for why we are allowed to do so is given by the first part of the next theorem.

Theorem 1. *Let k be a positive integer.*

1° *We have that $\mathcal{R}_c^{\text{inf-div}}(k) = \mathcal{E}_c(k)$ (equality of subsets of $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$). The target set of the bijection η from (1.5) can thus be regarded as $\mathcal{R}_c^{\text{inf-div}}(k)$.*

2° *There exists a bijection $\text{Reta} : \mathcal{R}_c(k) \rightarrow \mathcal{R}_c^{\text{inf-div}}(k)$ defined by the formula*

$$\text{Reta}(R_\mu) = \eta_\mu, \quad \forall \mu \in \mathcal{D}_c(k). \tag{1.6}$$

One has a purely combinatorial way of describing this bijection. More precisely, there exists an explicit summation formula which gives the coefficients of $\text{Reta}(f)$ in terms of the coefficients of f , where f is an arbitrary series in $\mathcal{R}_c(k)$. The summation formula is:

$$Cf_{(i_1, \dots, i_n)}(\text{Reta}(f)) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} Cf_{(i_1, \dots, i_n); \pi}(f), \tag{1.7}$$

$\forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k$. (The notations used in (1.7) for coefficients of power series are detailed in Definition 3.2 below. The partial order “ \ll ” on the set $NC(n)$ of non-crossing partitions of $\{1, \dots, n\}$ is discussed in Section 2 below, starting with Definition 2.5.)

3° *There exists a bijection $\mathbb{B} : \mathcal{D}_c(k) \rightarrow \mathcal{D}_c^{\text{inf-div}}(k)$ which is determined by the formula*

$$R_{\mathbb{B}(\mu)} = \eta_\mu, \quad \forall \mu \in \mathcal{D}_c(k). \tag{1.8}$$

In the case $k = 1$, \mathbb{B} coincides with the restriction of the Boolean Bercovici–Pata bijection to the set of compactly supported probability distributions on the real line.

When looking at the diagram (1.3), one can say that the map Reta is the “ R -transform incarnation” of the Boolean Bercovici–Pata bijection. It is remarkable that one can also describe Reta by the very explicit formulas (1.7) and (1.6) (where (1.6) is the one which suggested the name “Reta” – the transformation which “converts R to η ”).

We can supplement Theorem 1 with the following result, which is a k -dimensional version of the equivalence (2) \Leftrightarrow (3) in Theorem 6.1 of [4] (and thus provides a more in-depth explanation for why \mathbb{B} of Theorem 1 is indeed a k -dimensional version of the corresponding bijection from [4]).

Theorem 1’. *Let k be a positive integer. Let $(\mu_N)_{N=1}^\infty$ be a sequence of distributions in $\mathcal{D}_c(k)$, and let $p_1 < p_2 < \dots < p_N < \dots$ be a sequence of positive integers. Then the following two statements are equivalent:*

$$(1) \exists \lim_{N \rightarrow \infty} \underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N \text{ times}} =: \nu \in \mathcal{D}_c^{\text{inf-div}}(k),$$

$$(2) \exists \lim_{N \rightarrow \infty} \underbrace{\mu_N \uplus \dots \uplus \mu_N}_{p_N \text{ times}} =: \mu \in \mathcal{D}_c(k),$$

where the limits in (1), (2) are considered with respect to convergence in moments. Moreover, if (1) and (2) hold, then the resulting limits μ, ν are connected by the formula $\mathbb{B}(\mu) = \nu$, where \mathbb{B} is the bijection from Theorem 1.

We next proceed to presenting the second main result of this paper, which concerns a surprising property of the Boolean Bercovici–Pata bijection, in connection to the operation of free *multiplicative* convolution. This result takes place in a purely algebraic framework, and in order to present it we will move from $\mathcal{D}_c(k)$ to the larger set $\mathcal{D}_{\text{alg}}(k)$ of distributions of k -tuples in arbitrary (purely algebraic) non-commutative probability spaces. $\mathcal{D}_{\text{alg}}(k)$ consists in fact of *all* linear functionals $\mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C}$ which satisfy the normalization condition $\mu(1) = 1$.

The commutative diagram (1.3) has a “simplified” version living in the algebraic framework of $\mathcal{D}_{\text{alg}}(k)$. Indeed, for $\mu \in \mathcal{D}_{\text{alg}}(k)$ it is still possible to define the R -transform R_μ , and every series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ can be written uniquely as R_μ for some $\mu \in \mathcal{D}_{\text{alg}}(k)$. Moreover, every $\mu \in \mathcal{D}_{\text{alg}}(k)$ is (trivially) infinitely divisible in this purely algebraic framework; hence the two bijections displayed in (1.2) are now both replaced by the bijection

$$\mathcal{D}_{\text{alg}}(k) \ni \mu \mapsto R_\mu \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle. \quad (1.9)$$

On the other hand, for $\mu \in \mathcal{D}_{\text{alg}}(k)$ one can define the η -series η_μ , and every $f \in \mathbb{C}\langle\langle z_1, \dots, z_k \rangle\rangle$ can be written uniquely as η_μ for some $\mu \in \mathcal{D}_{\text{alg}}(k)$; so we also have a bijection

$$\mathcal{D}_{\text{alg}}(k) \ni \mu \mapsto \eta_\mu \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle, \quad (1.10)$$

which is the counterpart of the bijection from (1.5). This leads to the diagram

$$\begin{array}{ccc}
\mathcal{D}_{\text{alg}}(k) & \xrightarrow{\mathbb{B}} & \mathcal{D}_{\text{alg}}(k) \\
R \downarrow & \searrow \eta & \downarrow R \\
\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle & \xrightarrow{\text{Reta}} & \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle
\end{array} \tag{1.11}$$

where the vertical arrows are from (1.9), the diagonal is from (1.10), and the horizontal arrows \mathbb{B} and Reta are defined via the requirement that the diagram is commutative.

On the space $\mathcal{D}_{\text{alg}}(k)$ we can define an operation of *free multiplicative convolution* \boxtimes , as follows. Given $\mu, \nu \in \mathcal{D}_{\text{alg}}(k)$, one can always find random variables $x_1, \dots, x_k, y_1, \dots, y_k$ in a non-commutative probability space (\mathcal{M}, φ) such that the joint distribution of the k -tuple x_1, \dots, x_k is equal to μ , the joint distribution of the k -tuple y_1, \dots, y_k is equal to ν , and such that $\{x_1, \dots, x_k\}$ is freely independent from $\{y_1, \dots, y_k\}$ in (\mathcal{M}, φ) . The joint distribution of the k -tuple $x_1 y_1, \dots, x_k y_k$ turns out to depend only on μ and ν ; and the free multiplicative convolution $\mu \boxtimes \nu$ is equal, by definition, to the joint distribution of $x_1 y_1, \dots, x_k y_k$. (Note: what makes this definition not to work in the framework of $\mathcal{D}_c(k)$ is that, even if we assume that all of $x_1, \dots, x_k, y_1, \dots, y_k$ are selfadjoint elements in a C^* -probability space, the products $x_1 y_1, \dots, x_k y_k$ will no longer be selfadjoint, in general.)

By using this terminology, our second theorem is then stated as follows.

Theorem 2. *The bijection \mathbb{B} from the commutative diagram (1.11) is a homomorphism for \boxtimes . That is, we have*

$$\mathbb{B}(\mu \boxtimes \nu) = \mathbb{B}(\mu) \boxtimes \mathbb{B}(\nu), \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \tag{1.12}$$

It is also worth recording how Theorem 2 looks like when it is re-phrased in terms of R -transforms. This re-phrasing involves an operation \boxtimes , called *boxed convolution*, on the space of series $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. One way of defining \boxtimes is via the equation

$$R_{\mu \boxtimes \nu} = R_{\mu} \boxtimes R_{\nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \tag{1.13}$$

This equation says that \boxtimes is the “incarnation of \boxtimes ” obtained when one moves from $\mathcal{D}_{\text{alg}}(k)$ to $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ via the bijection (1.9). (Or, if we recall how \boxtimes is defined, we can say that the job of \boxtimes is to describe the multiplication of freely independent k -tuples, in terms of their R -transforms.) On the other hand, the operation \boxtimes can also be introduced in a purely combinatorial way – one has explicit formulas giving the coefficients of $f \boxtimes g$ in terms of the coefficients of f and of g , via summations over non-crossing partitions. The explicit formulas for the coefficients of $f \boxtimes g$ will be reviewed in the Section 7 of the paper; for more details on \boxtimes (including the explanation of why it is justified to use the name “convolution” for this operation) we refer to [8], Lectures 17 and 18.

The reformulation of Theorem 2 in terms of transforms goes as follows.

Theorem 2’. *The η -series satisfies the relation*

$$\eta_{\mu \boxtimes \nu} = \eta_{\mu} \boxstar \eta_{\nu}, \quad \forall \mu, \nu \in \mathcal{D}_{\text{alg}}(k). \quad (1.14)$$

The equivalence of Theorems 2 and 2’ is immediate. For instance if we assume Theorem 2’, then Theorem 2 is obtained as follows: for every $\mu, \nu \in \mathcal{D}_{\text{alg}}(k)$ we have that

$$\begin{aligned} R_{\mathbb{B}(\mu \boxtimes \nu)} &= \eta_{\mu \boxtimes \nu} \text{ (from diagram (1.11))} \\ &= \eta_{\mu} \boxstar \eta_{\nu} \text{ (by Theorem 2’)} \\ &= R_{\mathbb{B}(\mu)} \boxstar R_{\mathbb{B}(\nu)} \text{ (from diagram (1.11))} \\ &= R_{\mathbb{B}(\mu) \boxtimes \mathbb{B}(\nu)} \text{ (by Equation (1.13)).} \end{aligned}$$

So $\mathbb{B}(\mu \boxtimes \nu)$ and $\mathbb{B}(\mu) \boxtimes \mathbb{B}(\nu)$ have the same R-transform, and these two distributions must therefore be equal to each other.

It is interesting to compare the Equation (1.14) in Theorem 2’ with the quite similarly looking Equation (1.13) which precedes the theorem. We see here that the operation of boxed convolution \boxstar also pops up as the “incarnation of \boxtimes ” when one moves from $\mathcal{D}_{\text{alg}}(k)$ to $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ via the bijection (1.10), $\mu \mapsto \eta_{\mu}$, in lieu of the bijection $\mu \mapsto R_{\mu}$ from (1.9). (The bijection in (1.10) is quite a bit easier to work with than the one in (1.9) – see the discussion in Proposition 3.5 and Remark 3.6 below.)

Let us also mention here that in the case when $k = 1$, one of the usual ways of looking at \boxtimes is by viewing it as an operation on the set of probability distributions with support (not necessarily compact) contained in $[0, \infty)$. In this framework, a further discussion around the \boxtimes -multiplicativity of the bijection \mathbb{B} is made in [2] (by using complex analysis methods specific to the case $k = 1$, which also cover the situation of unbounded supports).

We conclude this introductory section by describing how the rest of the paper is organized. In the above discussion it was more relevant to consider first the framework of $\mathcal{D}_c(k)$, but for a more detailed presentation it is actually better to first clarify the simpler algebraic framework of $\mathcal{D}_{\text{alg}}(k)$. This is done in Section 3 of the paper, following to a review of some basic combinatorial structures done in Section 2. In Section 4 we give a more detailed introduction to $\mathcal{D}_c(k)$ and to the maps involved in the commutative diagram (1.3), and then in Section 5 we give the proofs of Theorems 1 and 1’. In Section 6 we return to the algebraic framework and present the result from the combinatorics of non-crossing partitions (Corollary 6.11) which lies at the core of our Theorems 2 and 2’. Finally, Section 7 is devoted to presenting the proofs of Theorems 2 and 2’.

2. Some basic combinatorial structures

The first part of this section gives a very concise review (intended mostly for setting notations) of non-crossing partitions. For a more detailed introduction to these partitions, and on how they are used in free probability, we refer to [8], Lectures 9 and 10.

2.1 Remark (review of $NC(n)$).

Let n be a positive integer and let $\pi = \{B_1, \dots, B_p\}$ be a partition of $\{1, \dots, n\}$ – i.e. B_1, \dots, B_p are pairwise disjoint non-void sets (called the **blocks** of π), and $B_1 \cup \dots \cup B_p = \{1, \dots, n\}$. We say that π is **non-crossing** if for every $1 \leq i < j < k < l \leq n$ such that i is in the same block with k and j is in the same block with l , it necessarily follows that all of i, j, k, l are in the same block of π . The set of all non-crossing partitions of $\{1, \dots, n\}$ will be denoted by $NC(n)$. On $NC(n)$ we consider the partial order given by **reversed refinement**: for $\pi, \rho \in NC(n)$, we write “ $\pi \leq \rho$ ” to mean that every block of ρ is a union of blocks of π . (In this paper we will use more than one partial order on $NC(n)$, but “ \leq ” will be always reserved for reversed refinement order.)

For $\pi \in NC(n)$, the number of blocks of π will be denoted by $|\pi|$. The minimal and maximal element of $(NC(n), \leq)$ are denoted by 0_n (the partition of $\{1, \dots, n\}$ into n singletons) and respectively 1_n (the partition of $\{1, \dots, n\}$ into one block).

A partition $\pi \in NC(n)$ has an **associated permutation** of $\{1, \dots, n\}$, which will be denoted by P_π . The permutation P_π is defined by the prescription that for every block $B = \{b_1, \dots, b_m\}$ of π , with $b_1 < \dots < b_m$, one creates a cycle of P_π , as follows:

$$P_\pi(b_1) = b_2, \dots, P_\pi(b_{m-1}) = b_m, P_\pi(b_m) = b_1.$$

2.2 Remark (review of the Kreweras complementation map).

This is a special order-reversing bijection $K : NC(n) \rightarrow NC(n)$. One way of describing how it works (which is actually the original definition from [5]) goes by using partitions of $\{1, \dots, 2n\}$.

Let π and ρ be two partitions of $\{1, \dots, n\}$. We will denote by

$$\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$$

the partition of $\{1, \dots, 2n\}$ which is obtained when one turns π into a partition of $\{1, 3, \dots, 2n-1\}$ and one turns ρ into a partition of $\{2, 4, \dots, 2n\}$, in the canonical way. That is, $\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ has blocks of the form $\{2a-1 \mid a \in A\}$ where A is a block of π , and has blocks of the form $\{2b \mid b \in B\}$ where B is a block of ρ .

A partition θ of $\{1, \dots, 2n\}$ is said to be **parity-preserving** if every block of θ either is contained in $\{1, 3, \dots, 2n-1\}$ or is contained in $\{2, 4, \dots, 2n\}$. The partitions of the form $\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ introduced above are parity-preserving; and conversely, every parity-preserving partition θ of $\{1, \dots, 2n\}$ is of the form $\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ for some uniquely determined partitions π, ρ of $\{1, \dots, n\}$.

The requirement that π and ρ are in $NC(n)$ is clearly necessary but not sufficient in order for $\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ to be in $NC(2n)$. If we fix $\pi \in NC(n)$ then the set

$$\{\rho \in NC(n) \mid \pi^{(\text{odd})} \sqcup \rho^{(\text{even})} \in NC(2n)\}$$

turns out to contain a largest partition ρ_{\max} , which is called the **Kreweras complement** of π and is denoted by $K(\pi)$. So $K(\pi)$ is defined by the requirement that for $\rho \in NC(n)$ we have:

$$\pi^{(\text{odd})} \sqcup \rho^{(\text{even})} \in NC(2n) \Leftrightarrow \rho \leq K(\pi). \quad (2.1)$$

It is easily verified that $\pi \mapsto K(\pi)$ is indeed an order-reversing bijection from $NC(n)$ to itself. Another feature of Kreweras complementation which is worth recording is that

$$|\pi| + |K(\pi)| = n + 1, \quad \forall \pi \in NC(n). \quad (2.2)$$

2.3 Remark (*Kreweras complementation via permutations*).

A convenient way of describing Kreweras complements is by using the permutations associated to non-crossing partitions. Indeed, the permutation $P_{K(\pi)}$ associated to the Kreweras complement of π turns out to be given by the neat formula

$$P_{K(\pi)} = P_\pi^{-1} P_{1_n}, \quad \pi \in NC(n). \quad (2.3)$$

(Note that the permutation P_{1_n} associated to the maximal partition $1_n \in NC(n)$ is just the cycle $1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$.)

The formula (2.3) can be extended in order to cover the concept of **relative Kreweras complement** of π in ρ , for $\pi, \rho \in NC(n)$ such that $\pi \leq \rho$. This relative Kreweras complement is a partition in $NC(n)$, which will be denoted by $K_\rho(\pi)$, and which is uniquely determined by the fact that the permutation associated to it is

$$P_{K_\rho(\pi)} = P_\pi^{-1} P_\rho. \quad (2.4)$$

Clearly, the Kreweras complementation map K discussed above is the relative complementation with respect to the maximal element 1_n of $NC(n)$.

It can be shown that, for a fixed $\rho \in NC(n)$, the map $\pi \mapsto K_\rho(\pi)$ is an order-reversing bijection from $\{\pi \in NC(n) \mid \pi \leq \rho\}$ onto itself. It can also be shown that

$$\pi \leq \rho_1 \leq \rho_2 \text{ in } NC(n) \Rightarrow K_{\rho_1}(\pi) \leq K_{\rho_2}(\pi). \quad (2.5)$$

For proofs of these facts, and for more details on relative Kreweras complements we refer to [8], Lecture 18.

Besides $NC(n)$, we will also use the partially ordered set of *interval partitions*.

2.4 Remark (*review of $\text{Int}(n)$*).

A partition π of $\{1, \dots, n\}$ is said to be an **interval partition** if every block B of π is of the form $B = [i, j] \cap \mathbb{Z}$ for some $1 \leq i \leq j \leq n$. The set of all interval partitions of $\{1, \dots, n\}$ will be denoted by $\text{Int}(n)$. It is clear that $\text{Int}(n) \subseteq NC(n)$, but it is in fact customary to view $(\text{Int}(n), \leq)$ as a partially ordered set on its own (where “ \leq ” still stands for the reversed refinement order on partitions). The enumeration arguments related to $\text{Int}(n)$ are often simplified by the fact that we have a natural bijection between $\text{Int}(n)$ and the collection $2^{\{1, \dots, n-1\}}$ of all subsets of $\{1, \dots, n-1\}$; this bijection maps $\pi \in \text{Int}(n)$ to the set

$$\left\{ m \mid 1 \leq m \leq n-1, \text{ and there exists a block } B \text{ of } \pi \text{ such that } \max(B) = m \right\}.$$

Moreover, this bijection is a poset isomorphism, if one endows $2^{\{1, \dots, n-1\}}$ with the partial order given by reversed inclusion.

We now move to introduce another partial order on $NC(n)$; this is not part of the usual lingo related to this topic, but will turn out to be essential for the developments shown in the present paper.

2.5 Definition. Let n be a positive integer, and let π and ρ be two partitions in $NC(n)$. We will write “ $\pi \ll \rho$ ” to mean that $\pi \leq \rho$ and that, in addition, the following condition is fulfilled:

$$\begin{cases} \text{For every block } C \text{ of } \rho \text{ there exists a block} \\ B \text{ of } \pi \text{ such that } \min(C), \max(C) \in B. \end{cases} \quad (2.6)$$

2.6 Remark. 1° Let $\pi, \rho \in NC(n)$ be such that $\pi \ll \rho$. Let C be a block of ρ , and let B be the block of π which contains $\min(C)$ and $\max(C)$. Then $B \subseteq C$ (because B has to be contained in a block of ρ , and this block can only be C), and we must have

$$\min(B) = \min(C), \quad \max(B) = \max(C). \quad (2.7)$$

2° It is immediately verified that \ll is indeed a partial order relation on $NC(n)$. It is much coarser than the reversed refinement order. For instance, the inequality $\pi \ll 1_n$ is not holding for all $\pi \in NC(n)$, but it rather amounts to the condition that the numbers 1 and n belong to the same block of π . At the other end of $NC(n)$, the inequality $\pi \gg 0_n$ can only take place when $\pi = 0_n$. (While looking at these trivial examples, let us also note that the partial order \ll does not generally behave well under taking Kreweras complements.)

3° Let $\rho = \{C_1, \dots, C_p\}$ be a fixed partition in $NC(n)$. For every $1 \leq q \leq p$ such that $|C_q| \geq 3$, let us split the block C_q into the doubleton $\{\min(C_q), \max(C_q)\}$ and $|C_q| - 2$ singletons; when doing this for all q we obtain a partition $\rho_0 \leq \rho$ in $NC(n)$, such that all the blocks of ρ_0 have either 1 or 2 elements. From Definition 2.5 it is clear that for $\pi \in NC(n)$ we have:

$$\pi \ll \rho \Leftrightarrow \rho_0 \leq \pi \leq \rho. \quad (2.8)$$

Consequently, the set $\{\pi \in NC(n) \mid \pi \ll \rho\}$ is just the interval $[\rho_0, \rho]$ (with respect to reversed refinement order) of $NC(n)$, and in order to describe it one can use the nice structure of such intervals of $NC(n)$ – as presented for instance in [8], Lecture 9.

4° Let π be a fixed partition in $NC(n)$. In contrast to what was observed in the preceding part of this remark, the set $\{\rho \in NC(n) \mid \rho \gg \pi\}$ isn't generally an interval with respect to reversed refinement order. This set has nevertheless nice enumerative properties, which will be described in Proposition 2.13 below. In Proposition 2.13 we will use a few basic facts concerning the nested structure of the blocks of a non-crossing partition, and we start by presenting these facts.

2.7 Definition. Let n be a positive integer and let A, B be two non-empty subsets of $\{1, \dots, n\}$. If $\min(A) \leq \min(B)$ and $\max(A) \geq \max(B)$, then we will say that A **embraces** B , and write $A \Subset B$.

2.8 Definition. 1° Let π be a partition in $NC(n)$, and let A be a block of π . If there is no block B of π such that $\min(B) < \min(A) \leq \max(A) < \max(B)$, then we say that A is an **outer block** of π .

2° For $\pi \in NC(n)$, the number of outer blocks of π will be denoted as $|\pi|_{\text{out}}$.

2.9 Remark. Let π be a partition in $NC(n)$.

1° It is clear that the set of blocks of π is partially ordered by embracing (where we stipulate that the block A is smaller than the block B with respect to this partial order if and only if $A \Subset B$). It is also clear that the outer blocks are precisely the minimal blocks of π with respect to this partial order.

2° Concerning the outer blocks of π , it is immediate that:

(a) The block of π which contains the number 1 is outer.

(b) If B is an outer block of π such that $\max(B) < n$, then there exists an outer block B' of π such that $\min(B') = \max(B) + 1$.

Hence if we denote $|\pi|_{\text{out}} =: r$ and if we list the outer blocks of π as B_1, B_2, \dots, B_r , in increasing order of their minimal elements, then we have

$$\min(B_1) = 1, \min(B_2) = \max(B_1) + 1, \dots, \min(B_r) = \max(B_{r-1}) + 1, \max(B_r) = n.$$

3° Let B_1, \dots, B_r be as in the preceding part of this remark, and let ρ be the interval partition with blocks $[\min(B_i), \max(B_i)] \cap \mathbb{Z}$, $1 \leq i \leq r$. It is immediate that $\rho \geq \pi$ and that ρ is the smallest interval partition (in the sense of reversed refinement order) which satisfies this inequality.

2.10 Proposition. Let π be a partition in $NC(n)$.

1° Let A and B be two distinct blocks of π . We have

$$A \Subset B \Leftrightarrow \left(\exists a_1, a_2 \in A \text{ and } b \in B \text{ such that } a_1 < b < a_2 \right). \quad (2.9)$$

2° Let A_1, A_2, B be blocks of π such that $A_1 \Subset B$ and $A_2 \Subset B$. Then either $A_1 \Subset A_2$ or $A_2 \Subset A_1$.

Proof. 1° is immediate, and left as exercise. For 2° we observe first that the intersection $[\min(A_1), \max(A_1)] \cap [\min(A_2), \max(A_2)]$ is non-empty, as it contains B . If it was not true that one of the intervals $[\min(A_1), \max(A_1)]$, $[\min(A_2), \max(A_2)]$ is contained in the other, we would obtain either that $\min(A_1) < \min(A_2) < \max(A_1) < \max(A_2)$ or that $\min(A_2) < \min(A_1) < \max(A_2) < \max(A_1)$, contradicting the fact that π is non-crossing.

QED

2.11 Remark. Let π be a partition in $NC(n)$. We consider the partial order given by embracing on the set of blocks of π , and we consider the so-called *Hasse diagram* for this partial order. The Hasse diagram is, by definition, a graph which has vertex set equal the set of blocks of π , and has an edge connecting two blocks $A_1 \neq A_2$ when one of them embraces the other (say for instance that $A_1 \Subset A_2$) and there is no third block of π which lies strictly between them (no $A \neq A_1, A_2$ such that $A_1 \Subset A \Subset A_2$). It is instructive to note that, as an immediate consequence of Proposition 2.10.2, the Hasse diagram we just described is a *forest* – that is, each of its connected components is a *tree* (a graph without circuits).

Let us also recall here the following concept: a forest is said to be *rooted* if one special vertex (a “root”) has been chosen in each of its connected components. There exists a natural way of rooting the above Hasse diagram because, as immediately seen, each of its connected components contains precisely one outer block of π . Thus we can view the Hasse diagram as a rooted forest, where the outer blocks are the roots.

We now return to the partial order \ll on $NC(n)$ that was introduced in Definition 2.5.

2.12 Remark. Let π, ρ be two partitions in $NC(n)$ such that $\pi \ll \rho$. Let C be a block of ρ and let B be the unique block of π which has $\min(B) = \min(C)$ and $\max(B) = \max(C)$

(cf Remark 2.6.1). It is an easy exercise, left to the reader, to check that

$$\left(\begin{array}{c} C \text{ is an outer} \\ \text{block of } \rho \end{array} \right) \Leftrightarrow \left(\begin{array}{c} B \text{ is an outer} \\ \text{block of } \pi \end{array} \right). \quad (2.10)$$

Moreover, it is easily seen that in this way we obtain a canonical one-to-one correspondence between the outer blocks of ρ and the outer blocks of π .

As a consequence of the above, we get that for a given $\pi \in NC(n)$, the number of blocks of any partition $\rho \in NC(n)$ such that $\rho \gg \pi$ is bounded by the inequalities

$$|\pi|_{\text{out}} \leq |\rho| \leq |\pi|. \quad (2.11)$$

(The first of the two inequalities holds because $|\pi|_{\text{out}} = |\rho|_{\text{out}} \leq |\rho|$, while the second one follows directly from the fact that $\rho \geq \pi$.)

2.13 Proposition. Let π be a partition in $NC(n)$. For every integer p satisfying $|\pi|_{\text{out}} \leq p \leq |\pi|$, we have that:

$$\text{card}\left\{\rho \in NC(n) \mid \rho \gg \pi \text{ and } |\rho| = p\right\} = \binom{|\pi| - |\pi|_{\text{out}}}{p - |\pi|_{\text{out}}}. \quad (2.12)$$

Proof. Consider the following two conditions which a set \mathfrak{M} of blocks of π may fulfil.

$$(C) \quad \left\{ \begin{array}{l} \text{“Convexity condition”. Whenever } A_1, A_2, A_3 \text{ are blocks of } \pi \text{ such} \\ \text{that } A_1 \subseteq A_2 \subseteq A_3 \text{ and such that } A_1, A_3 \in \mathfrak{M}, \text{ it follows} \\ \text{that } A_2 \in \mathfrak{M} \text{ as well.} \end{array} \right.$$

$$(Min) \quad \left\{ \begin{array}{l} \text{“Minimal element condition”. There exists a (necessarily unique)} \\ \text{block } B \in \mathfrak{M} \text{ such that } B \subseteq A, \forall A \in \mathfrak{M}. \end{array} \right.$$

Throughout this proof we will use the ad-hoc term of “CMin set of blocks” for a set \mathfrak{M} of blocks of π which fulfills both the conditions (C) and (Min). Moreover, suppose that $\mathfrak{M}_1, \dots, \mathfrak{M}_p$ are CMin sets of blocks of π such that $\mathfrak{M}_i \cap \mathfrak{M}_j = \emptyset$ for $i \neq j$, and such that $\mathfrak{M}_1 \cup \dots \cup \mathfrak{M}_p$ contains all blocks of π ; then we will refer to $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ by calling it a “CMin decomposition for the set of blocks of π ”.

The relevance of CMin sets of blocks in this proof comes from the following fact.

Fact 1. Let ρ be a partition in $NC(n)$ such that $\rho \gg \pi$, let C be a block of ρ , and denote $\mathfrak{M} := \{A \mid A \text{ block of } \pi, A \subseteq C\}$. Then \mathfrak{M} is a CMin set of blocks of π .

The verification of Fact 1 is immediate. For instance in order to verify that \mathfrak{M} satisfies the condition (C), one proceeds as follows. Let A_1, A_2, A_3 be blocks of π such that $A_1 \subseteq A_2 \subseteq A_3$ and such that $A_1, A_3 \in \mathfrak{M}$. Assume by contradiction that $A_2 \notin \mathfrak{M}$, hence that $A_2 \subseteq C'$ for some block C' of ρ where $C' \neq C$. From the hypothesis $A_1 \subseteq A_2 \subseteq A_3$ we deduce that $\min(A_1) < \min(A_2) < \min(A_3) < \max(A_2)$; since $\min(A_1), \min(A_3) \in C$ and $\min(A_2), \max(A_2) \in C'$, we have thus obtained a crossing between the blocks C and C' of ρ – contradiction.

An immediate consequence of Fact 1 is that we have:

Fact 2. Let $\rho = \{C_1, \dots, C_p\}$ be a partition in $NC(n)$ such that $\rho \gg \pi$, and for every $1 \leq i \leq p$ let us denote $\mathfrak{M}_i := \{A \mid A \text{ block of } \pi, A \subseteq C_i\}$. Then $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ is a CMin decomposition of the set of blocks of π .

On the other hand we have a converse of the Fact 2, stated as follows.

Fact 3. Let $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ be a CMin decomposition of the set of blocks of π , and for every $1 \leq i \leq p$ let us denote

$$C_i := \cup_{A \in \mathfrak{M}_i} A. \quad (2.13)$$

Consider the partition $\rho = \{C_1, \dots, C_p\}$ of $\{1, \dots, n\}$. Then $\rho \in NC(n)$, and $\rho \gg \pi$.

Verification of Fact 3. The only non-trivial point in the statement of Fact 3 is that the partition ρ is non-crossing. In order to verify this, let us fix $1 \leq a_1 < a_2 < a_3 < a_4 \leq n$ such that $a_1, a_3 \in C_i$ and $a_2, a_4 \in C_j$ for some $1 \leq i, j \leq p$. We want to prove that $i = j$.

Let A' and A'' denote the blocks of π which contain a_2 and a_3 , respectively. We have $A' \in \mathfrak{M}_j$ (because $A' \cap C_j \neq \emptyset$, hence A' must be contained in C_j), and $A'' \in \mathfrak{M}_i$ (by a similar argument). Consequently, we have the embracings

$$B' \Subset A' \text{ and } B'' \Subset A'', \quad (2.14)$$

where $B' \in \mathfrak{M}_j$ and $B'' \in \mathfrak{M}_i$ are the blocks of π which appear in the (Min) condition stated for \mathfrak{M}_j and for \mathfrak{M}_i , respectively. But let us observe that from the hypotheses given on a_1, a_2, a_3, a_4 , we can also infer that

$$B'' \Subset A' \text{ and } B' \Subset A''. \quad (2.15)$$

The embracings (2.15) are easily verified by using the criterion from Proposition 2.10.1; for instance for the first of the two embracings we observe that

$$\min(B'') = \min(C_i) \leq a_1 < a_2 < a_3 \leq \max(C_i) = \max(B''), \quad (2.16)$$

then we apply Proposition 2.10.1 to the situation where $\min(B'') < a_2 < \max(B'')$, with $a_2 \in A'$. (The equalities $\min(B'') = \min(C_i)$ and $\max(B'') = \max(C_i)$ appearing in (2.16) follow from how C_i is defined in (2.13), combined with the fact that $B'' \Subset A$ for every $A \in \mathfrak{M}_i$.)

From the embracings listed in (2.14) and (2.15), and by using Proposition 2.10.2, we find that we have either $B' \Subset B''$ or $B'' \Subset B'$. Say for instance that the first of these two possibilities takes place. Then we look at the embracings $B' \Subset B'' \Subset A'$ with $A', B' \in \mathfrak{M}_j$, and we use the convexity condition (C) stated for \mathfrak{M}_j , to obtain that $B'' \in \mathfrak{M}_j$. Hence $B'' \in \mathfrak{M}_i \cap \mathfrak{M}_j$, which implies the desired conclusion that $i = j$. (End of verification of Fact 3.)

It is clear that for any given integer p such that $|\pi|_{\text{out}} \leq p \leq |\pi|$, the Facts 2 and 3 together provide us with a bijection between $\{\rho \in NC(n) \mid \rho \gg \pi, |\rho| = p\}$ and the collection of all CMin decompositions $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ of the set of blocks of π . We will next observe that the latter CMin decompositions are in one-to-one correspondence with sets of blocks of π which contain all the $|\pi|_{\text{out}}$ outer blocks, plus $p - |\pi|_{\text{out}}$ non-outer blocks. The precise description of this (very natural) one-to-one correspondence is given in the next Fact

4. Since a set of $p - |\pi|_{\text{out}}$ non-outer blocks of π can be chosen in exactly $\binom{|\pi| - |\pi|_{\text{out}}}{p - |\pi|_{\text{out}}}$ ways, the discussion of Fact 4 will actually conclude the proof of the proposition.

Fact 4. (a) Let $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ be a CMin decomposition of the set of blocks of π , and for every $1 \leq i \leq p$ let B_i be the (uniquely determined) block of π which appears in the (Min) condition stated for \mathfrak{M}_i . Then $\{B_1, \dots, B_p\}$ is a set of blocks of π which contains all the outer blocks of π .

(b) Let $\{B_1, \dots, B_p\}$ be a set of blocks of π which contains all the outer blocks. There exists a unique CMin decomposition $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ of the set of blocks of π , such that $\{B_1, \dots, B_p\}$ is associated to $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ in the way described in (a) above.

The statement (a) in Fact 4 is trivial: an outer block B of π must satisfy $B = B_i$ for the unique $1 \leq i \leq p$ such that $B \in \mathfrak{M}_i$. The statement (b) is best understood from the perspective of the “rooted forest” framework discussed in Remark 2.11. (It is actually immediate to translate Fact 4 into a general statement about rooted forests, upon suitable interpretation for what “ \subseteq ” and “CMin” should mean in that framework.) We will indicate how $\{\mathfrak{M}_1, \dots, \mathfrak{M}_p\}$ is constructed by starting from $\{B_1, \dots, B_p\}$, and we will leave it as an exercise to the interested reader to fill in the details of this graph-theoretic argument.

So suppose that we are given a block A of π . The block A must be put into one of the sets of blocks $\mathfrak{M}_1, \dots, \mathfrak{M}_p$, and we have to indicate the procedure for finding the index $i \in \{1, \dots, p\}$ such that $A \in \mathfrak{M}_i$. We will describe this procedure by referring to the Hasse diagram discussed in Remark 2.11. Let B be the unique root (= outer block) which lies in the same connected component of the Hasse diagram as A . There exists a unique path from A to B in the Hasse diagram (this happens because the connected components of the Hasse diagram are trees). Let us denote this path as (A_0, A_1, \dots, A_s) , with $s \geq 0$, and where $A_0 = A$, $A_s = B$. Note that the path must intersect $\{B_1, \dots, B_p\}$ – indeed, we have in any case that $A_s \in \{B_1, \dots, B_p\}$, due to the hypothesis that $\{B_1, \dots, B_p\}$ contains all the outer blocks. Let r be the smallest number in $\{0, 1, \dots, s\}$ such that $A_r \in \{B_1, \dots, B_p\}$, and let i be the index in $\{1, \dots, p\}$ for which $A_r = B_i$. This i is the index we want – that is, the block A gets placed into the set of blocks \mathfrak{M}_i . **QED**

2.14 Remark. For a given partition $\pi \in NC(n)$, the total number of partitions $\rho \in NC(n)$ such that $\rho \gg \pi$ is equal to $2^{|\pi| - |\pi|_{\text{out}}}$. This is obtained by summing over p in Equation (2.12) of Proposition 2.13. Or at a “bijective” level, one can note that in the proof of Proposition 2.13 the partitions $\rho \in NC(n)$ such that $\rho \gg \pi$ end by being put into one-to-one correspondence with (arbitrarily chosen) sets of non-outer blocks of π .

3. R, η , Reta and \mathbb{B} , in the algebraic framework

Throughout this section we fix a positive integer k (the number of non-commuting indeterminates we are working with). We will deal with non-commutative distributions considered in an algebraic framework. The R and η series associated to such a distribution are reviewed in Definition 3.3, while Reta and \mathbb{B} are introduced in Definition 3.7.

3.1 Definition (non-commutative distributions).

1° We denote by $\mathbb{C}\langle X_1, \dots, X_k \rangle$ the algebra of non-commutative polynomials in X_1, \dots, X_k . Thus $\mathbb{C}\langle X_1, \dots, X_k \rangle$ has a linear basis

$$\{1\} \cup \{X_{i_1} \cdots X_{i_n} \mid n \geq 1, 1 \leq i_1, \dots, i_n \leq k\}, \quad (3.1)$$

where the monomials in the basis are multiplied by concatenation. When needed, $\mathbb{C}\langle X_1, \dots, X_k \rangle$ will be viewed as a $*$ -algebra, with $*$ -operation determined uniquely by the fact that each of X_1, \dots, X_k is selfadjoint.

2° Let (\mathcal{M}, φ) be a non-commutative probability space; that is, \mathcal{M} is a unital algebra over \mathbb{C} , and $\varphi : \mathcal{M} \rightarrow \mathbb{C}$ is a linear functional, normalized by the condition that $\varphi(1_{\mathcal{M}}) = 1$. For $x_1, \dots, x_k \in \mathcal{M}$, the **joint distribution** of x_1, \dots, x_k is the linear functional $\mu_{x_1, \dots, x_k} : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C}$ which acts on the linear basis (3.1) by the formula

$$\begin{cases} \mu_{x_1, \dots, x_k}(1) &= 1 \\ \mu_{x_1, \dots, x_k}(X_{i_1} \cdots X_{i_n}) &= \varphi(x_{i_1} \cdots x_{i_n}), \\ &\forall n \geq 1, 1 \leq i_1, \dots, i_n \leq k. \end{cases} \quad (3.2)$$

3° As already mentioned in the introduction, we will denote

$$\mathcal{D}_{\text{alg}}(k) := \{\mu : \mathbb{C}\langle X_1, \dots, X_k \rangle \rightarrow \mathbb{C} \mid \mu \text{ linear}, \mu(1) = 1\}. \quad (3.3)$$

Note that, unlike in the C^* -context, there is no positivity requirement in the definition of $\mathcal{D}_{\text{alg}}(k)$. It is immediate that $\mathcal{D}_{\text{alg}}(k)$ is precisely the set of linear functionals on $\mathbb{C}\langle X_1, \dots, X_k \rangle$ which can appear as joint distribution for some k -tuple x_1, \dots, x_k in a non-commutative probability space.

3.2 Definition (series and their coefficients).

1° Recall from the introduction that $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ denotes the space of power series with complex coefficients and with vanishing constant term, in k non-commuting indeterminates z_1, \dots, z_k . The general form of a series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ is thus

$$f(z_1, \dots, z_k) = \sum_{n=1}^{\infty} \sum_{i_1, \dots, i_n=1}^k \alpha_{(i_1, \dots, i_n)} z_{i_1} \cdots z_{i_n}, \quad (3.4)$$

where the coefficients $\alpha_{(i_1, \dots, i_n)}$ are from \mathbb{C} .

2° For $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$ we will denote by

$$\text{Cf}_{(i_1, \dots, i_n)} : \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \rightarrow \mathbb{C}$$

the linear functional which extracts the coefficient of $z_{i_1} \cdots z_{i_n}$ in a series $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. Thus for f written as in Equation (3.4) we have $\text{Cf}_{(i_1, \dots, i_n)}(f) = \alpha_{(i_1, \dots, i_n)}$.

3° Suppose we are given a positive integer n , some indices $i_1, \dots, i_n \in \{1, \dots, k\}$, and a partition $\pi \in NC(n)$. We define a (generally non-linear) functional

$$\text{Cf}_{(i_1, \dots, i_n); \pi} : \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \rightarrow \mathbb{C},$$

as follows. For every block $B = \{b_1, \dots, b_m\}$ of π , with $1 \leq b_1 < \dots < b_m \leq n$, let us use the notation

$$(i_1, \dots, i_n)|B := (i_{b_1}, \dots, i_{b_m}) \in \{1, \dots, k\}^m.$$

Then we define

$$\text{Cf}_{(i_1, \dots, i_n); \pi}(f) := \prod_{B \text{ block of } \pi} \text{Cf}_{(i_1, \dots, i_n)|B}(f), \quad \forall f \in \mathbb{C}_0 \langle \langle z_1, \dots, z_k \rangle \rangle. \quad (3.5)$$

(For example if we had $n = 5$ and $\pi = \{\{1, 4, 5\}, \{2, 3\}\}$, and if i_1, \dots, i_5 would be some fixed indices from $\{1, \dots, k\}$, then the above formula would become

$$\text{Cf}_{(i_1, i_2, i_3, i_4, i_5); \pi}(f) = \text{Cf}_{(i_1, i_4, i_5)}(f) \cdot \text{Cf}_{(i_2, i_3)}(f),$$

$f \in \mathbb{C}_0 \langle \langle z_1, \dots, z_k \rangle \rangle$.)

3.3 Definition (the series M, R, η).

Let μ be a distribution in $\mathcal{D}_{\text{alg}}(k)$. We will work with three series $M_\mu, R_\mu, \eta_\mu \in \mathbb{C}_0 \langle \langle z_1, \dots, z_k \rangle \rangle$ that are associated to μ , and are defined as follows.

1° The **moment series** of μ will be denoted by M_μ . Its coefficients are defined by

$$\text{Cf}_{(i_1, \dots, i_n)}(M_\mu) = \mu(X_{i_1} \cdots X_{i_n}), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k.$$

2° The **R-transform** of μ will be denoted by R_μ . The coefficients of R_μ are defined by a formula which expresses them as polynomial expressions in the coefficients of M_μ :

$$\text{Cf}_{(i_1, \dots, i_n)}(R_\mu) = \sum_{\pi \in NC(n)} s(\pi) \cdot \text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k, \quad (3.6)$$

where on the right-hand side of (3.6) we used the notation for generalized coefficients from Definition 3.2.3, and where $\{s(\pi) \mid \pi \in \cup_{n=1}^\infty NC(n)\}$ is a special family of coefficients (not depending on μ). For a given $\pi \in NC(n)$, the precise description of $s(\pi)$ goes as follows: consider the Kreweras complement $K(\pi) = \{B_1, \dots, B_p\} \in NC(n)$, and define $s(\pi) := s_{|B_1|} \cdots s_{|B_p|}$, where the s_m are signed Catalan numbers, $s_m = (-1)^{m-1}(2m - 2)!/(m - 1)!m!$ for $m \geq 1$.

The explicit description of the coefficients $s(\pi)$ is probably less illuminating than explaining that they appear in the following way. The Equations in (3.6) are equivalent to another family of equations of the same form, where the roles of M_μ and R_μ are switched (that is, the coefficients of M_μ are written as polynomial expressions in the coefficients of R_μ). The $s(\pi)$ are chosen such that in this equivalent family of equations we only have plain summations:

$$\text{Cf}_{(i_1, \dots, i_n)}(M_\mu) = \sum_{\pi \in NC(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(R_\mu), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k. \quad (3.7)$$

3° The **η -series** of μ will be denoted by η_μ . The procedure for defining η_μ in terms of M_μ is analogous to the one used for defining R_μ , only that now we are using the set $\text{Int}(n)$ of interval partitions instead of $NC(n)$. The precise formula giving the coefficients of η_μ is

$$\text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu) = \sum_{\pi \in \text{Int}(n)} (-1)^{1+|\pi|} \text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k. \quad (3.8)$$

The choice of the values “ ± 1 ” on the right-hand side of (3.8) is made so that the reverse connection between the coefficients of M_μ and of η_μ is described by plain summations:

$$\text{Cf}_{(i_1, \dots, i_n)}(M_\mu) = \sum_{\pi \in \text{Int}(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(\eta_\mu), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k. \quad (3.9)$$

3.4 Remark. 1° The formulas connecting the moment series M_μ to the series R_μ and η_μ are well-known, and are usually stated as relations between certain multi-linear functionals (moment functionals and cumulant functionals) on non-commutative probability spaces. More precisely, the fomulas for R_μ relate to the concept of *free cumulant functionals* introduced in [9], while the formulas for η_μ relate to the *Boolean cumulant functionals* which go all the way back to [15].

2° It is clear that $\mu \mapsto M_\mu$ is a bijection from $\mathcal{D}_{\text{alg}}(k)$ onto $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. Since the formulas which define R_μ and η_μ in terms of M_μ are reversible (in the way explained in the parts 2° and 3° of the above definition), it is immediate that $\mu \mapsto R_\mu$ and $\mu \mapsto \eta_\mu$ also are bijections from $\mathcal{D}_{\text{alg}}(k)$ onto $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$; these are the bijections displayed in (1.9) and (1.10) of the introduction.

3° Let μ be a distribution in $\mathcal{D}_{\text{alg}}(k)$. The Equation (3.7) describing the passage from R_μ to M_μ has a straightforward extension to a summation formula which gives the *generalized* coefficients of M_μ in terms of those of R_μ . This formula is

$$\text{Cf}_{(i_1, \dots, i_n); \rho}(M_\mu) = \sum_{\substack{\pi \in NC(n), \\ \pi \leq \rho}} \text{Cf}_{(i_1, \dots, i_n); \pi}(R_\mu), \quad (3.10)$$

holding for any $\rho \in NC(n)$ (and where the original formula (3.7) corresponds to the case when $\rho = 1_n$).

A similar statement holds in connection to the passage from η_μ to M_μ – one obtains a summation formula which gives the generalized coefficients of M_μ in terms of those of η_μ , extending Equation (3.9).

4° In the analytic theory of distributions of 1 variable, the definition of the η -series of a probability measure μ on \mathbb{R} appears usually as “ $\eta = \Psi/(1 + \Psi)$ ”, where Ψ is defined by an integral formula and corresponds, in the case when μ has compact support, to the moment series of μ – see for instance the presentation at the beginning of Section 2 of [1]. The next proposition shows that such an approach can be also used in our multi-variable setting.

3.5 Proposition. Let μ be a distribution in $\mathcal{D}_{\text{alg}}(k)$. We have

$$\eta_\mu = M_\mu / (1 + M_\mu), \quad (3.11)$$

where the division on the right-hand side of (3.11) stands for the commuting product $M_\mu(1 + M_\mu)^{-1}$ in the ring $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. Conversely, M_μ can be obtained from η_μ by the formula

$$M_\mu = \eta_\mu / (1 - \eta_\mu). \quad (3.12)$$

Proof. We will verify the relation

$$M_\mu = \eta_\mu + M_\mu \cdot \eta_\mu, \quad (3.13)$$

out of which (3.11) and (3.12) follow via easy algebraic manipulations.

We will fix for the whole proof some integers $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$, and we will verify the equality of the coefficients of $z_{i_1} z_{i_2} \cdots z_{i_n}$ in the series on the two sides of (3.13). Our computations will rely on the immediate observation that

$$\text{Int}(n) = \{1_n\} \cup \text{Int}^{(1)}(n) \cup \cdots \cup \text{Int}^{(n-1)}(n),$$

disjoint union, where for $1 \leq m \leq n-1$ we denote

$$\text{Int}^{(m)}(n) := \{\pi \in \text{Int}(n) \mid \{m+1, \dots, n\} \text{ is a block of } \pi\}.$$

We will also use the obvious fact that for every $1 \leq m \leq n-1$ we have a natural bijection $\text{Int}^{(m)}(n) \ni \pi \mapsto \pi' \in \text{Int}(m)$, where π' is obtained from π by removing its right-most block $\{m+1, \dots, n\}$.

So then, compute:

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(M_\mu) &= \sum_{\pi \in \text{Int}(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(\eta_\mu) \quad (\text{by Equation (3.9)}) \\ &= \text{Cf}_{(i_1, \dots, i_n); 1_n}(\eta_\mu) + \sum_{m=1}^{n-1} \left(\sum_{\pi \in \text{Int}^{(m)}(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(\eta_\mu) \right) \\ &= \text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu) + \sum_{m=1}^{n-1} \left(\sum_{\pi' \in \text{Int}(m)} \text{Cf}_{(i_1, \dots, i_m); \pi'}(\eta_\mu) \right) \cdot \text{Cf}_{(i_{m+1}, \dots, i_n)}(\eta_\mu) \\ &= \text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu) + \sum_{m=1}^{n-1} \text{Cf}_{(i_1, \dots, i_m)}(M_\mu) \cdot \text{Cf}_{(i_{m+1}, \dots, i_n)}(\eta_\mu) \quad (\text{by Equation (3.9)}) \\ &= \text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu + M_\mu \cdot \eta_\mu), \end{aligned}$$

as required. **QED**

3.6 Remark. Since our presentation in this section emphasizes the parallelism between R and η , let us briefly mention that there exists a counterpart for Equation (3.11) in the theory of the R -transform – but this is a more complicated, implicit equation involving M_μ and R_μ . This latter equation is not used in the present paper (for a presentation of how it looks and how it is derived, we refer to [8], Lecture 16).

3.7 Definition. Refer to the bijections R and η from $\mathcal{D}_{\text{alg}}(k)$ onto $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ which were observed in Remark 3.4.2. We define two new bijections:

$$\mathbb{B} := R^{-1} \circ \eta : \mathcal{D}_{\text{alg}}(k) \rightarrow \mathcal{D}_{\text{alg}}(k), \quad (3.14)$$

and

$$\text{Reta} := \eta \circ R^{-1} : \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \rightarrow \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle. \quad (3.15)$$

3.8 Remark. ^{1°} We have now formally defined all the bijections which appear in the commutative diagram (1.11) from the introduction. (The commutativity of this diagram is ensured by the very definition of \mathbb{B} and Reta .)

^{2°} As explained in the introduction, \mathbb{B} stands for “Boolean Bercovici–Pata bijection”, while Reta gets its name from the formula

$$\text{Reta}(R_\mu) = \eta_\mu, \quad \forall \mu \in \mathcal{D}_{\text{alg}}(k) \quad (3.16)$$

(it is the transformation of $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ which “converts R to η ”).

3° We will next prove that Reta can also be described by an explicit formula via summations over non-crossing partitions. This will be the same formula as indicated in Equation (1.7) of Theorem 1, with the difference that we will now state and prove the formula for an arbitrary $f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$, rather than just for series in the smaller set $\mathcal{R}_c(k) \subseteq \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. We mention that in the particular case when $k = 1$, some formulas equivalent to (3.17) and (3.18) of the next proposition have appeared in [7] (cf Equation (4.10) and the proof of Equation (5.1) in that paper).

3.9 Proposition. Let f, g be series in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ such that $\text{Reta}(f) = g$. Then:

1° For every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$ we have

$$\text{Cf}_{(i_1, \dots, i_n)}(g) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \text{Cf}_{(i_1, \dots, i_n); \pi}(f). \quad (3.17)$$

2° For every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$ we have

$$\text{Cf}_{(i_1, \dots, i_n)}(f) = \sum_{\substack{\rho \in NC(n), \\ \rho \ll 1_n}} (-1)^{1+|\rho|} \text{Cf}_{(i_1, \dots, i_n); \rho}(g). \quad (3.18)$$

Proof. By the definition of Reta, there exists a distribution $\mu \in \mathcal{D}_{\text{alg}}(k)$ such that $R_\mu = f$, $\eta_\mu = g$.

1° We calculate as follows:

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(g) &= \text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu) \\ &= \sum_{\rho \in \text{Int}(n)} (-1)^{1+|\rho|} \text{Cf}_{(i_1, \dots, i_n); \rho}(M_\mu) \quad (\text{by Equation (3.8)}) \\ &= \sum_{\rho \in \text{Int}(n)} (-1)^{1+|\rho|} \left(\sum_{\substack{\pi \in NC(n), \\ \pi \leq \rho}} \text{Cf}_{(i_1, \dots, i_n); \pi}(R_\mu) \right) \quad (\text{by Equation (3.10)}) \\ &= \sum_{\pi \in NC(n)} \left(\sum_{\substack{\rho \in \text{Int}(n), \\ \rho \geq \pi}} (-1)^{1+|\rho|} \right) \text{Cf}_{(i_1, \dots, i_n); \pi}(f), \end{aligned} \quad (3.19)$$

where at the last equality sign we performed a change in the order of summation.

Now let us fix for the moment a partition $\pi \in NC(n)$. We claim that

$$\sum_{\substack{\rho \in \text{Int}(n), \\ \rho \geq \pi}} (-1)^{1+|\rho|} = \begin{cases} 1 & \text{if } \pi \ll 1_n \\ 0 & \text{otherwise.} \end{cases} \quad (3.20)$$

Indeed, the set $\{\rho \in \text{Int}(n) \mid \rho \geq \pi\}$ has a smallest element $\tilde{\rho}$, with $|\tilde{\rho}| = |\rho|_{\text{out}} =: r$ (cf Remark 2.9.3). Hence we have

$$\begin{aligned} \sum_{\substack{\rho \in \text{Int}(n), \\ \rho \geq \pi}} (-1)^{1+|\rho|} &= \sum_{\substack{\rho \in \text{Int}(n), \\ \rho \geq \tilde{\rho}}} (-1)^{1+|\rho|} = \sum_{m=1}^r \binom{r-1}{m-1} \cdot (-1)^{m+1} \\ &= (1-1)^{r-1} = \begin{cases} 1 & \text{if } |\rho|_{\text{out}} = 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } \pi \ll 1_n \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

as required, where during the calculation we used the immediate fact that for every $1 \leq m \leq r$ there are $\binom{r-1}{m-1}$ partitions $\rho \in \text{Int}(n)$ such that $\rho \geq \tilde{\rho}$ and $|\rho| = m$.

The formula (3.17) now follows, when (3.20) is substituted in (3.19).

2^o Before starting on the calculation which leads to (3.18), let us note that it is straightforward to extend the Equation (3.17) proved in 1^o to a formula expressing a *generalized* coefficient of g in terms of the generalized coefficients of f . (This is analogous to how (3.7) was extended to (3.10) in Remark 3.4.3.) The precise formula extending (3.17) is

$$\text{Cf}_{(i_1, \dots, i_n); \rho}(g) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll \rho}} \text{Cf}_{(i_1, \dots, i_n); \pi}(f), \quad (3.21)$$

holding for an arbitrary $\rho \in NC(n)$, and where the original Equation (3.17) corresponds to the case $\rho = 1_n$.

We now start from the right-hand side of (3.18), and substitute $\text{Cf}_{(i_1, \dots, i_n); \rho}(g)$ in terms of generalized coefficients of f , as indicated by Equation (3.21). We get:

$$\sum_{\substack{\rho \in NC(n), \\ \rho \ll 1_n}} (-1)^{1+|\rho|} \left(\sum_{\substack{\pi \in NC(n), \\ \pi \ll \rho}} \text{Cf}_{(i_1, \dots, i_n); \pi}(f) \right);$$

after changing the order of summation over π and ρ , this becomes

$$\sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \left(\sum_{\substack{\rho \in NC(n) \\ \text{such that} \\ \pi \ll \rho \ll 1_n}} (-1)^{1+|\rho|} \right) \cdot \text{Cf}_{(i_1, \dots, i_n); \pi}(f). \quad (3.22)$$

The sum over ρ which appears in (3.22) can be treated exactly as we did with (3.20), but where this time instead of a counting argument in $\text{Int}(n)$ we now invoke Proposition 2.13. The reader should have no difficulty to verify that what we get is the following: for a fixed $\pi \in NC(n)$ such that $\pi \ll 1_n$, we have

$$\sum_{\substack{\rho \in NC(n) \\ \text{such that} \\ \pi \ll \rho \ll 1_n}} (-1)^{1+|\rho|} = \begin{cases} 1 & \text{if } \pi = 1_n \\ 0 & \text{otherwise.} \end{cases}$$

Substituting this in (3.22) leads to the conclusion that the expression considered there is equal to $\text{Cf}_{(i_1, \dots, i_n)}(f)$, as required. **QED**

4. $\mathcal{D}_c(k)$, and infinite divisibility with respect to \boxplus and \boxplus

In this section, k is a fixed positive integer.

4.1 Definition. 1° Refer to the space $\mathcal{D}_{\text{alg}}(k)$ introduced in Definition 3.1.3. We denote

$$\mathcal{D}_c(k) = \left\{ \mu \in \mathcal{D}_{\text{alg}}(k) \left| \begin{array}{l} \exists C^*\text{-probability space } (\mathcal{M}, \varphi) \\ \text{and selfadjoint elements } x_1, \dots, x_k \in \mathcal{M} \\ \text{such that } \mu_{x_1, \dots, x_k} = \mu \end{array} \right. \right\} \quad (4.1)$$

(where the fact that (\mathcal{M}, φ) is a C^* -probability space means that \mathcal{M} is a unital C^* -algebra, and that $\varphi : \mathcal{M} \rightarrow \mathbb{C}$ is a positive linear functional such that $\varphi(1_{\mathcal{M}}) = 1$).

The notation “ $\mathcal{D}_c(k)$ ” is chosen to remind of “distributions with compact support” – indeed, in the case when $k = 1$ we have a natural identification between $\mathcal{D}_c(1)$ and the set of probability distributions with compact support on \mathbb{R} .

2° Refer to the bijections $R, \eta : \mathcal{D}_{\text{alg}}(k) \rightarrow \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ from Remark 3.4.2. We will denote

$$\mathcal{R}_c(k) := \{f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \mid \exists \mu \in \mathcal{D}_c(k) \text{ such that } R_\mu = f\} \quad (4.2)$$

and respectively

$$\mathcal{E}_c(k) := \{f \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \mid \exists \mu \in \mathcal{D}_c(k) \text{ such that } \eta_\mu = f\}. \quad (4.3)$$

We thus have bijections

$$\mathcal{D}_c(k) \ni \mu \mapsto R_\mu \in \mathcal{R}_c(k) \text{ and } \mathcal{D}_c(k) \ni \mu \mapsto \eta_\mu \in \mathcal{E}_c(k),$$

as indicated in (1.2) and (1.5) of the introduction section.

Clearly, the set of distributions $\mathcal{D}_c(k)$ is much smaller than $\mathcal{D}_{\text{alg}}(k)$. In what follows we will use the characterization of $\mathcal{D}_c(k)$ given by the next proposition. This characterization is most likely a “folklore” fact; for the reader’s convenience, we include an outline of the proof.

4.2 Proposition. Let μ be a functional in $\mathcal{D}_{\text{alg}}(k)$. Then μ belongs to $\mathcal{D}_c(k)$ if and only if it satisfies the following two conditions:

- (i) $\mu(P^*P) \geq 0, \forall P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$.
- (ii) There exists a constant $\gamma > 0$ such that

$$|\mu(X_{i_1} \cdots X_{i_n})| \leq \gamma^n, \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k. \quad (4.4)$$

Proof. The necessity of the conditions (i) and (ii) is immediate, and left as exercise. We will outline the argument for their sufficiency. The proof is of course a variation of the

GNS construction, the only special point that has to be addressed is the boundedness of the left multiplication operators.

So suppose that μ satisfies (i) and (ii). The positivity condition (i) allows us to create a Hilbert space \mathcal{H} and a linear map $\mathbb{C}\langle X_1, \dots, X_k \rangle \ni P \mapsto \widehat{P} \in \mathcal{H}$, such that the image of this map is a dense subspace of \mathcal{H} , and such that the inner product on \mathcal{H} is determined by the formula

$$\langle \widehat{P}, \widehat{Q} \rangle = \mu(Q^*P), \quad \forall P, Q \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

By using the boundedness condition (ii) we will prove the inequality

$$\|\widehat{X_i P}\| \leq \gamma \|\widehat{P}\|, \quad \forall 1 \leq i \leq k, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_k \rangle, \quad (4.5)$$

where $\gamma > 0$ is the constant appearing in (ii). This amounts to proving that

$$\mu(P^* X_i^2 P) \leq \gamma^2 \mu(P^* P), \quad \forall 1 \leq i \leq k, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_k \rangle. \quad (4.6)$$

We will obtain (4.6) by a repeated application of the Cauchy–Schwarz inequality in \mathcal{H} , which says that

$$|\mu(P^* Q)| \leq \mu(P^* P)^{1/2} \mu(Q^* Q)^{1/2}, \quad \forall P, Q \in \mathbb{C}\langle X_1, \dots, X_k \rangle. \quad (4.7)$$

So let us fix $i \in \{1, \dots, k\}$ and $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ and let us use (4.7) with $Q = X_i^2 P$. We obtain

$$\mu(P^* X_i^2 P) \leq \mu(P^* P)^{1/2} \mu(P^* X_i^4 P)^{1/2}. \quad (4.8)$$

Then let us use (4.7) once again, but this time with $Q = X_i^4 P$. We get (after taking both sides to power 1/2) that

$$\mu(P^* X_i^4 P)^{1/2} \leq \mu(P^* P)^{1/4} \mu(P^* X_i^8 P)^{1/4};$$

and replacing the latter inequality into (4.8) leads to

$$\mu(P^* X_i^2 P) \leq \mu(P^* P)^{3/4} \mu(P^* X_i^8 P)^{1/4}. \quad (4.9)$$

It is immediate how this trick can be iterated (use Cauchy–Schwarz with $Q = X_i^8 P$, then with $Q = X_i^{16} P$, etc), to obtain that

$$\mu(P^* X_i^2 P) \leq \mu(P^* P)^{(2^n - 1)/2^n} \mu(P^* X_i^{2^{n+1}} P)^{1/2^n}, \quad \forall n \geq 1. \quad (4.10)$$

We can now use the condition (ii) to get an upper bound on the factor $\mu(P^* X_i^{2^{n+1}} P)^{1/2^n}$ on the right-hand side of the inequality (4.10). Indeed, let us write $P = \sum_{j=1}^m \alpha_j P_j$, where for every $1 \leq j \leq m$ we have that α_j is in $\mathbb{C} \setminus \{0\}$ and that P_j a monomial of length l_j in the variables X_1, \dots, X_k . Then

$$P^* X_i^{2^{n+1}} P = \sum_{j, j'=1}^m \overline{\alpha_j} \alpha_{j'} (P_j^* X_i^{2^{n+1}} P_{j'}),$$

which implies that

$$\begin{aligned} \mu(P^* X_i^{2^{n+1}} P) &\leq \sum_{j, j'=1}^m |\alpha_j \alpha_{j'}| \cdot |\mu(P_j^* X_i^{2^{n+1}} P_{j'})| \\ &\leq \sum_{j, j'=1}^m |\alpha_j \alpha_{j'}| \cdot \gamma^{l_j + 2^{n+1} + l_{j'}} \quad (\text{by condition (ii)}) \\ &= \gamma^{2^{n+1}} C, \end{aligned}$$

where $C := \sum_{j,j'=1}^m |\alpha_j \alpha_{j'}| \cdot \gamma^{l_j + l_{j'}}$ is a constant which depends only on P (but not on n). We thus obtain that the right-hand side of (4.10) is bounded from above by

$$\mu(P^*P)^{(2^n-1)/2^n} \cdot \left(\gamma^{2^{n+1}} C \right)^{1/2^n},$$

and (4.6) follows when we let $n \rightarrow \infty$.

Finally, by using (4.5) it is immediatly seen that one can define a family of bounded linear operators $T_1, \dots, T_k \in B(\mathcal{H})$, determined by the formula

$$T_i \widehat{P} = \widehat{X_i P}, \quad \forall 1 \leq i \leq k, \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_k \rangle.$$

We leave it as an easy exercise to the reader to check that $T_i = T_i^*$, $1 \leq i \leq k$, and that if one considers the C^* -probability space $(B(\mathcal{H}), \varphi)$ with $\varphi(T) := \langle T \widehat{1}, \widehat{1} \rangle$, $T \in B(\mathcal{H})$, then the joint distribution of T_1, \dots, T_k in $(B(\mathcal{H}), \varphi)$ is precisely the functional μ that we started with. It follows that $\mu \in \mathcal{D}_c(k)$, as required. **QED**

We now come to the operations \boxplus and \boxplus that were mentioned in the introduction. It will be convenient to consider them in the larger algebraic framework provided by the space $\mathcal{D}_{\text{alg}}(k)$. Each of these two operations has its own theory, developped in connection to a form of independence for non-commutative random variables. We will briefly comment on this in the Remark 4.4 below, but we will not need to go into details about non-commutative independence. Indeed, for the approach used in this paper (where the R -transform and η -series play the main role), we can simply regard \boxplus and \boxplus as the binary operations that are linearized by R and η , respectively.

4.3 Definition. Let μ and ν be distributions in $\mathcal{D}_{\text{alg}}(k)$.

1° The **free additive convolution** $\mu \boxplus \nu$ is the unique distribution in $\mathcal{D}_{\text{alg}}(k)$ which has R -transform equal to

$$R_{\mu \boxplus \nu} = R_\mu + R_\nu. \quad (4.11)$$

2° The **Boolean convolution** $\mu \boxplus \nu$ is the unique distribution in $\mathcal{D}_{\text{alg}}(k)$ which has η -series equal to

$$\eta_{\mu \boxplus \nu} = \eta_\mu + \eta_\nu. \quad (4.12)$$

4.4 Remark. The above definition reverses the order of how things are usually considered in the literature – usually \boxplus and \boxplus are considered first, and then R and η appear as linearizing transforms for these two operations. The way how \boxplus and \boxplus are usually considered is in connection to the concepts of *free independence* and respectively *Boolean independence* for subsets of a non-commutative probability space (\mathcal{M}, φ) . More precisely, suppose that we have elements $x_1, \dots, x_k, y_1, \dots, y_k \in \mathcal{M}$ such that the joint distribution of x_1, \dots, x_k is equal to μ , and the joint distribution of y_1, \dots, y_k is equal to ν . If $\{x_1, \dots, x_k\}$ is freely independent from $\{y_1, \dots, y_k\}$ in (\mathcal{M}, φ) , then the joint distribution of the k -tuple $x_1 + y_1, \dots, x_k + y_k$ is equal to $\mu \boxplus \nu$; while if $\{x_1, \dots, x_k\}$ is Boolean independent from $\{y_1, \dots, y_k\}$ in (\mathcal{M}, φ) , then the joint distribution of the k -tuple $x_1 + y_1, \dots, x_k + y_k$ is equal to $\mu \boxplus \nu$.

In this paper we will not need to review the precise definitions of free and of Boolean independence. We need however to mention one fact about \boxplus and \boxplus which comes out of

the approach via non-commutative independence, namely that:

$$\mu, \nu \in \mathcal{D}_c(k) \implies \mu \boxplus \nu, \mu \uplus \nu \in \mathcal{D}_c(k). \quad (4.13)$$

Indeed, if $\mu, \nu \in \mathcal{D}_c(k)$, then it can be shown that $x_1, \dots, x_k, y_1, \dots, y_k$ from the preceding paragraph can always be found to be selfadjoint elements in a C^* -probability space. Since the elements $x_1 + y_1, \dots, x_k + y_k$ will then also be selfadjoint, it follows that the convolutions $\mu \boxplus \nu$ and $\mu \uplus \nu$ are still in $\mathcal{D}_c(k)$. Thus we can (and will) also view \boxplus and \uplus as binary operations on $\mathcal{D}_c(k)$.

Let us also record here that, as a consequence of (4.13) and of the Equations (4.11) and (4.12) in Definition 4.3, it is immediate that the sets of series $\mathcal{R}_c(k), \mathcal{E}_c(k) \subseteq \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ are closed under addition.

We now move to discuss infinite divisibility. We discuss first the case of \boxplus .

4.5 Definition. 1° Let μ be in $\mathcal{D}_c(k)$. If for every positive integer N there exists a distribution $\mu_N \in \mathcal{D}_c(k)$ such that

$$\underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{N \text{ times}} = \mu,$$

then we will say that μ is **\boxplus -infinitely divisible**. The set of all distributions $\mu \in \mathcal{D}_c(k)$ which are \boxplus -infinitely divisible will be denoted by $\mathcal{D}_c^{\text{inf-div}}(k)$.

2° We denote

$$\mathcal{R}_c^{\text{inf-div}}(k) = \{f \in \mathcal{R}_c(k) \mid f = R_\mu \text{ for a distribution } \mu \in \mathcal{D}_c^{\text{inf-div}}(k)\}. \quad (4.14)$$

4.6 Remark. Infinite divisibility with respect to \boxplus relates to how $\mathcal{R}_c(k)$ behaves under the operation of multiplication by scalars from $(0, \infty)$. Let us record here that we have:

$$(f \in \mathcal{R}_c(k), t \in [1, \infty)) \implies tf \in \mathcal{R}_c(k). \quad (4.15)$$

This is a non-trivial fact, which appears in connection to how R -transforms behave under compressions by free projections – see Lecture 14 of [8] for more details.

On the other hand, $\mathcal{R}_c(k)$ is not closed under multiplication by scalars from $(0, 1)$. For a fixed series $f \in \mathcal{R}_c(k)$ we have in fact that

$$\begin{aligned} tf \in \mathcal{R}_c(k), \forall t \in (0, 1) &\Leftrightarrow \frac{1}{N}f \in \mathcal{R}_c(k), \forall N \in \mathbb{N} \setminus \{0\} \\ &\Leftrightarrow f \in \mathcal{R}_c^{\text{inf-div}}(k), \end{aligned}$$

where the first of these equivalences follows from (4.15), and the second one is a direct consequence of Definition 4.5.

4.7 Remark. Following the above considerations about infinite divisibility for \boxplus , it would be now natural to do the parallel discussion and introduce the corresponding notations for \uplus . But it turns out that no new notations are needed, as $\mathcal{E}_c(k)$ is closed under multiplication by scalars from $(0, \infty)$ (and consequently, all the distributions in $\mathcal{D}_c(k)$ are

\oplus -infinitely divisible). This fact is proved in the next proposition, by using an operator model for how to achieve the multiplication of an η -series by a scalar $t \in (0, 1)$. We mention here that in the case $k = 1$ another proof of this proposition can be given by using complex analysis methods (specific to the case $k = 1$ only); see Theorem 3.6 of [10]. To our knowledge, the case $k \geq 2$ was not treated before (it is e.g mentioned as an open problem in the recent thesis [16]).

4.8 Proposition. If $f \in \mathcal{E}_c(k)$ and $t \in (0, \infty)$, then $tf \in \mathcal{E}_c(k)$.

Proof. Since we know that $\mathcal{E}_c(k)$ is closed under addition, it suffices to do the case when $t \in (0, 1)$. We fix for the whole proof a series $f \in \mathcal{E}_c(k)$ and a number $t \in (0, 1)$. We denote by μ the unique distribution in $\mathcal{D}_c(k)$ such that $\eta_\mu = f$; the goal of the proof is to find a distribution $\nu \in \mathcal{D}_c(k)$ such that $\eta_\nu = tf$.

Let x_1, \dots, x_k be selfadjoint elements in a C^* -probability space (\mathcal{M}, φ) such that the joint distribution of x_1, \dots, x_k is equal to μ . By considering the GNS representation of φ we may assume, without loss of generality, that $\mathcal{M} = B(\mathcal{H})$ for a Hilbert space \mathcal{H} , and that φ is the vector-state given by a unit vector $\xi_o \in \mathcal{H}$ (that is, $\varphi(x) = \langle x\xi_o, \xi_o \rangle$ for every $x \in B(\mathcal{H})$).

Let us consider a new Hilbert space

$$\mathcal{K} := \mathcal{V} \oplus (\mathcal{H} \ominus \mathbb{C}\xi_o) \oplus (\mathcal{H} \ominus \mathbb{C}\xi_o), \quad (4.16)$$

where \mathcal{V} is a Hilbert space of dimension 2, spanned by two vectors Ω_1, Ω_2 such that

$$\|\Omega_1\| = 1 = \|\Omega_2\|, \quad \langle \Omega_1, \Omega_2 \rangle = t - 1. \quad (4.17)$$

We consider moreover two isometric operators $J_1, J_2 : \mathcal{H} \rightarrow \mathcal{K}$, defined by

$$\begin{cases} J_1(\alpha\xi_o + \xi) &= \alpha\Omega_1 \oplus \xi \oplus 0 \\ J_2(\alpha\xi_o + \xi) &= \alpha\Omega_2 \oplus 0 \oplus \xi, \quad \forall \alpha \in \mathbb{C}, \forall \xi \in \mathcal{H} \ominus \mathbb{C}\xi_o. \end{cases} \quad (4.18)$$

It is immediately verified that the adjoints $J_1^*, J_2^* \in B(\mathcal{K}, \mathcal{H})$ are described by the formula

$$\begin{cases} J_1^*(v \oplus \xi_1 \oplus \xi_2) &= \langle v, \Omega_1 \rangle \xi_o + \xi_1 \\ J_2^*(v \oplus \xi_1 \oplus \xi_2) &= \langle v, \Omega_2 \rangle \xi_o + \xi_2, \quad \forall \alpha \in \mathbb{C}, \forall \xi_1, \xi_2 \in \mathcal{H} \ominus \mathbb{C}\xi_o. \end{cases} \quad (4.19)$$

As a consequence of (4.17)–(4.19), we have

$$J_1^* J_2 = J_2^* J_1 = (t - 1)P_o, \quad (4.20)$$

where $P_o \in B(\mathcal{H})$ is the orthogonal projection onto the 1-dimensional space $\mathbb{C}\xi_o$ (that is, $P_o\xi = \langle \xi, \xi_o \rangle \xi_o$ for every $\xi \in B(\mathcal{H})$).

Consider the operators $y_1, \dots, y_k \in B(\mathcal{K})$ defined by

$$y_i := J_1 x_i J_1^* + J_2 x_i J_2^*, \quad 1 \leq i \leq k. \quad (4.21)$$

Let $\nu \in \mathcal{D}_c(k)$ be the joint distribution of y_1, \dots, y_k in the C^* -probability space $(B(\mathcal{K}), \psi)$, where ψ is the vector-state given on $B(\mathcal{K})$ by the unit-vector Ω defined as

$$\Omega := \frac{1}{\sqrt{2t}} \left((\Omega_1 + \Omega_2) \oplus 0 \oplus 0 \right) \in \mathcal{K}. \quad (4.22)$$

We want to obtain an explicit formula for the coefficients of the moment series M_ν . So let us fix a positive integer n and some indices $i_1, \dots, i_n \in \{1, \dots, k\}$, and let us compute:

$$\begin{aligned}
\text{Cf}_{(i_1, \dots, i_n)}(M_\nu) &= \psi(y_{i_1} \cdots y_{i_n}) = \langle y_{i_1} \cdots y_{i_n} \Omega, \Omega \rangle \quad (\text{by the def. of } \nu \text{ and of } \psi) \\
&= \langle (J_1 x_{i_1} J_1^* + J_2 x_{i_1} J_2^*) \cdots (J_1 x_{i_n} J_1^* + J_2 x_{i_n} J_2^*) \Omega, \Omega \rangle \quad (\text{by the def. of } y_1, \dots, y_k) \\
&= \sum_{r_1, \dots, r_n=1}^2 \langle (J_{r_1} x_{i_1} J_{r_1}^*) \cdots (J_{r_n} x_{i_n} J_{r_n}^*) \Omega, \Omega \rangle \\
&= \sum_{r_1, \dots, r_n=1}^2 \langle x_{i_1} (J_{r_1}^* J_{r_2}) x_{i_2} \cdots (J_{r_{n-1}}^* J_{r_n}) x_{i_n} (J_{r_n}^* \Omega), (J_{r_1}^* \Omega) \rangle \\
&= \frac{t}{2} \sum_{r_1, \dots, r_n=1}^2 \langle x_{i_1} (J_{r_1}^* J_{r_2}) x_{i_2} \cdots (J_{r_{n-1}}^* J_{r_n}) x_{i_n} \xi_o, \xi_o \rangle, \tag{4.23}
\end{aligned}$$

where at the last equality sign we took into account that (as is immediately verified) $J_1^* \Omega = J_2^* \Omega = \sqrt{t/2} \xi_o$. The next thing to be taken into account is that, for $r, r' \in \{1, 2\}$, we have:

$$J_r^* J_{r'} = \begin{cases} 1_{B(\mathcal{H})} & \text{if } r = r' \\ (t-1)P_o & \text{if } r \neq r'. \end{cases}$$

So for any n -tuple $(r_1, \dots, r_n) \in \{1, 2\}^n$, the operator

$$x_{i_1} (J_{r_1}^* J_{r_2}) x_{i_2} (J_{r_2}^* J_{r_3}) \cdots (J_{r_{n-1}}^* J_{r_n}) x_{i_n} \in B(\mathcal{H}) \tag{4.24}$$

really depends only on the set of positions $m \in \{1, \dots, n-1\}$ where we have $r_m \neq r_{m+1}$. If we write this set of positions as $\{m_1, \dots, m_p\}$ with $1 \leq m_1 < m_2 < \dots < m_p \leq n-1$, then the product considered in (4.24) equals

$$(x_{i_1} \cdots x_{i_{m_1}}) ((t-1)P_o) (x_{i_{m_1+1}} \cdots x_{i_{m_2}}) ((t-1)P_o) \cdots ((t-1)P_o) (x_{i_{m_p+1}} \cdots x_{i_n}). \tag{4.25}$$

It is moreover immediately seen that when one applies the vector-state $\varphi = \langle \cdot, \xi_o, \xi_o \rangle$ to the operator in (4.25), the result is

$$(t-1)^p \langle x_{i_1} \cdots x_{i_{m_1}} \xi_o, \xi_o \rangle \langle x_{i_{m_1+1}} \cdots x_{i_{m_2}} \xi_o, \xi_o \rangle \cdots \langle x_{i_{m_p+1}} \cdots x_{i_n} \xi_o, \xi_o \rangle. \tag{4.26}$$

(Note: It is not ruled out that the set $\{m \mid 1 \leq m \leq n-1, r_m \neq r_{m+1}\}$ could be empty. The formula (4.26) still holds in this case, the quantity appearing there being then equal to $\langle x_{i_1} \cdots x_{i_n} \xi_o, \xi_o \rangle$.)

It is convenient that in the calculations shown in the preceding paragraph we encode the sequence $1 \leq m_1 < \dots < m_p \leq n-1$ by the interval partition $\pi = \{B_1, \dots, B_{p+1}\}$ where $B_1 = [1, m_1] \cap \mathbb{Z}$, $B_2 = (m_1, m_2] \cap \mathbb{Z}$, \dots , $B_p = (m_{p-1}, m_p] \cap \mathbb{Z}$, $B_{p+1} = (m_p, n] \cap \mathbb{Z}$. (In the case when $\{m_1, \dots, m_p\} = \emptyset$, we take π to be 1_n , the partition with only one block.) The quantity in (4.26) then becomes

$$(t-1)^{|\pi|-1} \text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu), \tag{4.27}$$

where the generalized coefficient $\text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu)$ is considered in the sense of Definition 3.2.3. Note moreover that for every given partition $\pi \in \text{Int}(n)$ there are exactly two n -tuples $(r_1, \dots, r_n) \in \{1, 2\}^n$ for which the set $\{m \mid 1 \leq m \leq n-1, r_m \neq r_{m+1}\}$ is encoded by π (one of these two n -tuples has $r_1 = 1$, and the other has $r_1 = 2$).

If we now return to the expression in (4.23), and if in that summation formula we replace n -tuples $(r_1, \dots, r_n) \in \{1, 2\}^n$ by partitions $\pi \in \text{Int}(n)$, then we obtain:

$$\text{Cf}_{(i_1, \dots, i_n)}(M_\nu) = t \sum_{\pi \in \text{Int}(n)} (t-1)^{|\pi|-1} \text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu).$$

It is more suggestive to re-write the above equation in the form

$$\text{Cf}_{(i_1, \dots, i_n)}\left(\frac{t-1}{t}M_\nu\right) = \sum_{\pi \in \text{Int}(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}\left((t-1)M_\mu\right), \quad (4.28)$$

holding for every $n \geq 1$ and every $1 \leq i_1, \dots, i_n \leq k$. From (4.28) we see that the series $((t-1)/t)M_\nu$ is obtained from $(t-1)M_\mu$ by exactly the formula expressing a moment series in terms of the corresponding η -series – cf Equation (3.9) in Definition 3.3. But we saw in Proposition 3.5 how the latter formula can be written in a compressed way, in terms of the series themselves rather than in terms of coefficients. Applied to the situation at hand, Proposition 3.5 will thus give us that

$$\frac{t-1}{t}M_\nu = \frac{(t-1)M_\mu}{1 - (t-1)M_\mu}. \quad (4.29)$$

Finally, we use Equation (4.29) in order to compute η_ν . We leave it as a straightforward exercise to the reader to check that when we write $\eta_\nu = M_\nu/(1 + M_\nu)$ and then replace M_ν in terms of M_μ by using (4.29), what comes out is simply that $\eta_\nu = (tM_\mu)/(1 + M_\mu) = t\eta_\mu = tf$, as we wanted. **QED**

5. Proofs of Theorems 1 and 1'

Throughout this section, k is a fixed positive integer.

5.1 Remark. Theorems 1 and 1' take place in the framework of $\mathcal{D}_c(k)$, but in their proofs it will be nevertheless useful to rely on occasion on the larger algebraic framework provided by $\mathcal{D}_{\text{alg}}(k)$. For example: when we need to construct a distribution in $\mathcal{D}_c(k)$ which satisfies certain requirements, it may come in handy to first observe a distribution $\mu \in \mathcal{D}_{\text{alg}}(k)$ which satisfies the given requirements, and then to verify (by using Proposition 4.2) that μ belongs in fact to the subset $\mathcal{D}_c(k) \subseteq \mathcal{D}_{\text{alg}}(k)$.

In connection to the above, it will be convenient to place the next definition (for convergence of sequences) in the larger framework of $\mathcal{D}_{\text{alg}}(k)$.

5.2 Definition. 1° Let μ and $(\mu_N)_{N=1}^\infty$ be distributions in $\mathcal{D}_{\text{alg}}(k)$. The notation “ $\lim_{N \rightarrow \infty} \mu_N = \mu$ ” will be used to mean that $(\mu_N)_{N=1}^\infty$ **converges in moments** to μ , i.e. that

$$\lim_{N \rightarrow \infty} \mu_N(P) = \mu(P), \quad \forall P \in \mathbb{C}\langle X_1, \dots, X_k \rangle. \quad (5.1)$$

2° Let f and $(f_N)_{N=1}^\infty$ be series in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. The notation “ $\lim_{N \rightarrow \infty} f_N = f$ ” will be used to mean that $(f_N)_{N=1}^\infty$ **converges coefficientwise** to f , i.e. that

$$\lim_{N \rightarrow \infty} \text{Cf}_{(i_1, \dots, i_n)}(f_N) = \text{Cf}_{(i_1, \dots, i_n)}(f), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k. \quad (5.2)$$

We now start on a sequence of lemmas which will gradually build towards the statements of Theorems 1 and 1'.

5.3 Lemma. Let μ and $(\mu_N)_{N=1}^\infty$ be distributions in $\mathcal{D}_{\text{alg}}(k)$. The following three statements are equivalent:

$$(1) \lim_{N \rightarrow \infty} \mu_N = \mu; \quad (2) \lim_{N \rightarrow \infty} R_{\mu_N} = R_\mu; \quad (3) \lim_{N \rightarrow \infty} \eta_{\mu_N} = \eta_\mu.$$

Proof. It is immediate that the convergence in moments from (1) is equivalent to a statement (1') referring to the coefficientwise convergence of the corresponding moment series,

$$(1') \lim_{N \rightarrow \infty} M_{\mu_N} = M_\mu.$$

On the other hand, it is immediate that we have (1') \Leftrightarrow (2) and (1') \Leftrightarrow (3), due to the explicit formulas relating the coefficients of the series M, R, η via (finite!) summations over partitions, as presented in Definition 3.3 above. **QED**

5.4 Lemma. Let f and $(f_N)_{N=1}^\infty$ be series in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$, such that $\lim_{N \rightarrow \infty} f_N = f$, and let $(t_N)_{N=1}^\infty$ be a sequence in $(0, \infty)$ such that $\lim_{N \rightarrow \infty} t_N = \infty$. For every $N \geq 1$ let us consider the series

$$g_N := t_N \cdot \text{Reta}\left(\frac{1}{t_N} f_N\right) \quad \text{and} \quad h_N := t_N \cdot \text{Reta}^{-1}\left(\frac{1}{t_N} f_N\right)$$

in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. Then $\lim_{N \rightarrow \infty} g_N = f$ and $\lim_{N \rightarrow \infty} h_N = f$.

Proof. For every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$ we have:

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(g_N) &= t_N \cdot \text{Cf}_{(i_1, \dots, i_n)}\left(\text{Reta}\left(\frac{1}{t_N} f_N\right)\right) \\ &= t_N \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} \text{Cf}_{(i_1, \dots, i_n); \pi}\left(\frac{1}{t_N} f_N\right) \quad (\text{by Proposition 3.9.1}) \\ &= \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} t_N^{1-|\pi|} \text{Cf}_{(i_1, \dots, i_n); \pi}(f_N), \end{aligned} \tag{5.3}$$

where at the last equality sign we used the obvious homogeneity property of $\text{Cf}_{(i_1, \dots, i_n); \pi}$. When we make $N \rightarrow \infty$ in (5.3), the only term which survives is the one corresponding to $\pi = 1_n$, and it follows that

$$\lim_{N \rightarrow \infty} \text{Cf}_{(i_1, \dots, i_n)}(g_N) = \lim_{N \rightarrow \infty} \text{Cf}_{(i_1, \dots, i_n)}(f_N) = \text{Cf}_{(i_1, \dots, i_n)}(f).$$

This proves that $\lim_{N \rightarrow \infty} g_N = f$. The argument for $\lim_{N \rightarrow \infty} h_N = f$ is similar, with the only difference that we now use Proposition 3.9.2 instead of 3.9.1. **QED**

5.5 Lemma. Let $(\mu_N)_{N=1}^\infty$ be in $\mathcal{D}_{\text{alg}}(k)$ and let $p_1 < p_2 < \dots < p_N < \dots$ be a sequence of positive integers.

1° Suppose there exists $\mu \in \mathcal{D}_{\text{alg}}(k)$ such that $\lim_{N \rightarrow \infty} \underbrace{\mu_N \uplus \dots \uplus \mu_N}_{p_N} = \mu$. Then it follows that $\lim_{N \rightarrow \infty} \underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N} = \mathbb{B}(\mu)$ (where $\mathbb{B} : \mathcal{D}_{\text{alg}}(k) \rightarrow \mathcal{D}_{\text{alg}}(k)$ is the bijection from Definition 3.7).

2° Suppose there exists $\nu \in \mathcal{D}_{\text{alg}}(k)$ such that $\lim_{N \rightarrow \infty} \underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N} = \nu$. Then it follows that $\lim_{N \rightarrow \infty} \underbrace{\mu_N \uplus \dots \uplus \mu_N}_{p_N} = \mathbb{B}^{-1}(\nu)$.

Proof. 1° Let us denote

$$f_N := \underbrace{\eta_{\mu_N \uplus \dots \uplus \mu_N}}_{p_N} = p_N \cdot \eta_{\mu_N}, \quad N \geq 1,$$

and let $f := \eta_\mu$. From Lemma 5.3 and the given hypothesis it follows that $\lim_{N \rightarrow \infty} f_N = f$. Thus if we let

$$h_N := p_N \cdot \text{Reta}^{-1}\left(\frac{1}{p_N} f_N\right), \quad N \geq 1,$$

then Lemma 5.4 gives us that $\lim_{N \rightarrow \infty} h_N = f$ as well. But let us observe that, for every $N \geq 1$:

$$\begin{aligned} h_N &= p_N \cdot \text{Reta}^{-1}(\eta_{\mu_N}) \quad (\text{since } f_N = p_N \cdot \eta_{\mu_N}) \\ &= p_N \cdot R_{\mu_N} \quad (\text{by definition of Reta, Def. 3.7}) \\ &= R_{\underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N}}. \end{aligned}$$

On the other hand we can write $f = \eta_\mu = R_{\mathbb{B}(\mu)}$ (by definition of \mathbb{B}). So the convergence $\lim_{N \rightarrow \infty} h_N = f$ amounts in fact to

$$\lim_{N \rightarrow \infty} R_{\underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N}} = R_{\mathbb{B}(\mu)},$$

and the conclusion that $\lim_{N \rightarrow \infty} \underbrace{\mu_N \boxplus \dots \boxplus \mu_N}_{p_N} = \mathbb{B}(\mu)$ follows from Lemma 5.3.

2° The proof of this statement is identical to the proof of 1°, where now we switch the roles of \boxplus and \uplus , the roles of R and η , and we use the other part of Lemma 5.4. **QED**

5.6 Lemma. Let μ be a distribution in $\mathcal{D}_{\text{alg}}(k)$, and consider the series $R_\mu, \eta_\mu \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. The following statements are equivalent:

(1) There exists a constant $\gamma > 0$ such that

$$|\mu(X_{i_1} \cdots X_{i_n})| \leq \gamma^n, \quad \forall n \geq 1, \quad \forall 1 \leq i_1, \dots, i_n \leq k.$$

(2) There exists a constant $\gamma > 0$ such that

$$|\text{Cf}_{(i_1, \dots, i_n)}(R_\mu)| \leq \gamma^n, \quad \forall n \geq 1, \quad \forall 1 \leq i_1, \dots, i_n \leq k.$$

(3) There exists a constant $\gamma > 0$ such that

$$|\text{Cf}_{(i_1, \dots, i_n)}(\eta_\mu)| \leq \gamma^n, \quad \forall n \geq 1, \quad \forall 1 \leq i_1, \dots, i_n \leq k.$$

Proof. Both the equivalences (1) \Leftrightarrow (2) and (1) \Leftrightarrow (3) follow from the explicit relations via summations over partitions which connect the coefficients of the series M_μ, R_μ, η_μ , where one uses suitable bounds for how many terms there are in the summations, and for the size of the coefficients (if there are any coefficients involved). For example, when proving that (1) \Rightarrow (2), one uses the bound

$$\text{card}(NC(n)) = \frac{(2n)!}{n!(n+1)!} \leq 4^n, \quad \forall n \geq 1,$$

and one also uses the fact (easily proved by induction) that the constants “ $s(\pi)$ ” appearing in Equation (3.6) from Definition 3.3.2 satisfy

$$|s(\pi)| \leq 4^n, \quad \forall n \geq 1, \quad \forall \pi \in NC(n).$$

Suppose $\gamma > 0$ is such that (1) holds. Then the coefficients of the moment series M_μ satisfy $|\text{Cf}_{(i_1, \dots, i_n)}(M_\mu)| \leq \gamma^n$, for every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$, and more generally

$$|\text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu)| \leq \gamma^n, \quad \forall n \geq 1, \quad \forall 1 \leq i_1, \dots, i_n \leq k, \quad \forall \pi \in NC(n).$$

So then for every $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$ we have

$$|\text{Cf}_{(i_1, \dots, i_n)}(R_\mu)| = \left| \sum_{\pi \in NC(n)} s(\pi) \cdot \text{Cf}_{(i_1, \dots, i_n); \pi}(M_\mu) \right| \leq \sum_{\pi \in NC(n)} 4^n \cdot \gamma^n \leq (16\gamma)^n,$$

and (2) holds, with γ replaced by 16γ . The arguments for (2) \Rightarrow (1) and for (1) \Leftrightarrow (3) are similar, and require in fact smaller corrections for γ . **QED**

5.7 Lemma. We have $\mathcal{R}_c^{\text{inf-div}}(k) = \mathcal{E}_c(k)$, where the subsets $\mathcal{R}_c^{\text{inf-div}}(k)$ and $\mathcal{E}_c(k)$ of $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ are as in Definition 4.5.2 and in Definition 4.1.2, respectively.

Proof. “ \subseteq ” Let f be a series in $\mathcal{R}_c^{\text{inf-div}}(k)$, about which we want to show that $f \in \mathcal{E}_c(k)$. Let μ be the unique distribution in $\mathcal{D}_{\text{alg}}(k)$ such that $\eta_\mu = f$; proving that $f \in \mathcal{E}_c(k)$ is equivalent to proving that $\mu \in \mathcal{D}_c(k)$. We will prove the latter fact, by verifying that μ satisfies the conditions (i) and (ii) from Proposition 4.2.

The verification of (ii) is immediate, in view of Lemma 5.6. Indeed, since $f \in \mathcal{R}_c^{\text{inf-div}}(k) \subseteq \mathcal{R}_c(k)$, we know there exists a distribution $\nu \in \mathcal{D}_c(k)$ such that $R_\nu = f$. By using the condition (ii) for ν and the equivalence (1) \Leftrightarrow (2) in Lemma 5.6, we find that there exists $\gamma > 0$ such that $|\text{Cf}_{(i_1, \dots, i_n)}(f)| \leq \gamma^n$ for every $n \geq 1$ and every $1 \leq i_1, \dots, i_n \leq k$. But then we can use the fact that $f = \eta_\mu$ and the equivalence (1) \Leftrightarrow (3) in Lemma 5.6, to obtain that μ satisfies (ii).

In order to verify that μ satisfies condition (i), we proceed as follows. For every $N \geq 1$ let us consider the series

$$g_N := N \cdot \text{Reta}\left(\frac{1}{N}f\right) \in \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle, \quad (5.4)$$

and the unique distribution $\mu_N \in \mathcal{D}_{\text{alg}}(k)$ such that $\eta_{\mu_N} = g_N$. We have $\lim_{N \rightarrow \infty} g_N = f$, by Lemma 5.4. This convergence can also be written as $\lim_{N \rightarrow \infty} \eta_{\mu_N} = \eta_\mu$, and it gives us that $\lim_{N \rightarrow \infty} \mu_N = \mu$, by Lemma 5.3. But now let us observe that for every $N \geq 1$ we have

$$\begin{aligned} & \frac{1}{N}f \in \mathcal{R}_c(k) \quad (\text{because } f \in \mathcal{R}_c^{\text{inf-div}}(k)) \\ \Rightarrow & \text{Reta}\left(\frac{1}{N}f\right) \in \mathcal{E}_c(k) \quad (\text{because Reta maps } \mathcal{R}_c(k) \text{ onto } \mathcal{E}_c(k)) \\ \Rightarrow & g_N = N \cdot \text{Reta}\left(\frac{1}{N}f\right) \in \mathcal{E}_c(k) \quad (\text{because } \mathcal{E}_c(k) \text{ is closed under addition}) \\ \Rightarrow & \mu_N \in \mathcal{D}_c(k) \quad (\text{by the definition of } \mathcal{E}_c(k)). \end{aligned}$$

In particular it follows that every μ_N satisfies condition (i), and then it is clear that the limit in moments $\mu = \lim_{N \rightarrow \infty} \mu_N$ has to satisfy (i) too.

“ \supseteq ” Let us observe that it suffices to prove the weaker inclusion

$$\mathcal{E}_c(k) \subseteq \mathcal{R}_c(k). \quad (5.5)$$

Indeed, if (5.5) is known, then for an arbitrary series $f \in \mathcal{E}_c(k)$ we get that:

$$\begin{aligned} & tf \in \mathcal{E}_c(k), \quad \forall t \in (0, 1) \quad (\text{by Proposition 4.8}) \\ \Rightarrow & tf \in \mathcal{R}_c(k) \quad \forall t \in (0, 1) \quad (\text{by (5.5)}) \\ \Rightarrow & f \in \mathcal{R}_c^{\text{inf-div}}(k) \quad (\text{by Remark 4.6}). \end{aligned}$$

Hence for this part of the proof it suffices if we fix a series $f \in \mathcal{E}_c(k)$, and prove that $f \in \mathcal{R}_c(k)$. The argument for this is pretty much identical to the one shown above, in the proof of the inclusion \subseteq . That is, we consider the unique distribution $\nu \in \mathcal{D}_{\text{alg}}(k)$ such that $R_\nu = f$, and we prove that $\nu \in \mathcal{D}_c(k)$, by verifying that it satisfies the conditions (i) and (ii) from Proposition 4.2. The verification of (ii) proceeds exactly as in the proof of \subseteq (we look at the distribution $\mu \in \mathcal{D}_c(k)$ which has $\eta_\mu = f = R_\nu$, and we use Lemma 5.6 twice, in connection to μ , f and ν). The verification of (i) also proceeds on the same lines as shown in the proof of \subseteq , with the difference that instead of the series g_N from (5.4) we now look at

$$h_N := N \cdot \text{Reta}^{-1}\left(\frac{1}{N}f\right), \quad N \geq 1, \quad (5.6)$$

and we consider the distributions $(\nu_N)_{N=1}^\infty$ in $\mathcal{D}_{\text{alg}}(k)$ which have $R_{\nu_N} = h_N$, $N \geq 1$. We leave it as an exercise to the reader to adjust the argument shown in the proof of \subseteq in order to verify that $\nu_N \in \mathcal{D}_c(k)$ for every $N \geq 1$, and that $\lim_{N \rightarrow \infty} \nu_N = \nu$. The property (i) for ν is therefore obtained by passing to the limit the property (i) for the ν_N . **QED**

5.8 Remark (proofs of Theorems 1 and 1’).

At this moment we are in fact only left to observe that all the statements made in Theorems 1 and 1’ are covered by the arguments shown above, as follows.

(a) Part 1^o of Theorem 1 is covered by Lemma 5.7.

(b) For part 2^o of Theorem 1, we observe that by its very definition (and by the definitions of $\mathcal{R}_c(k)$ and $\mathcal{E}_c(k)$), the bijection $\text{Reta} : \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle \rightarrow \mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$ from Definition 3.7 maps $\mathcal{R}_c(k)$ onto $\mathcal{E}_c(k)$. Since $\mathcal{E}_c(k) = \mathcal{R}_c^{\text{inf-div}}(k)$, we get that Reta is indeed

a bijection between $\mathcal{R}_c(k)$ and $\mathcal{R}_c^{\text{inf-div}}(k)$. The explicit formula given for Reta in Equation (1.7) was proved in Proposition 3.9.

(c) The map $\mathbb{B} : \mathcal{D}_c(k) \rightarrow \mathcal{D}_c^{\text{inf-div}}(k)$ in part 3^o of Theorem 1 is the composition of the bijections $\eta : \mathcal{D}_c(k) \rightarrow \mathcal{R}_c^{\text{inf-div}}(k)$ and $R^{-1} : \mathcal{R}_c^{\text{inf-div}}(k) \rightarrow \mathcal{D}_c^{\text{inf-div}}(k)$, and is therefore itself bijective. It is also clear that $\mathbb{B} : \mathcal{D}_c(k) \rightarrow \mathcal{D}_c^{\text{inf-div}}(k)$ is the restriction of the bijection $\mathbb{B} : \mathcal{D}_{\text{alg}}(k) \rightarrow \mathcal{D}_{\text{alg}}(k)$ from Definition 3.7.

(d) Theorem 1' follows immediately from Lemma 5.5 and the fact, recorded above, that the algebraic bijection $\mathbb{B} : \mathcal{D}_{\text{alg}}(k) \rightarrow \mathcal{D}_{\text{alg}}(k)$ maps $\mathcal{D}_c(k)$ onto $\mathcal{D}_c^{\text{inf-div}}(k)$.

(e) The last thing left is the compatibility (stated in part 3^o of Theorem 1) of \mathbb{B} with the corresponding bijection from [4]. This is an immediate consequence of Theorem 1', which was in fact given as a multi-variable counterpart for the corresponding statement in [4].

5.9 Remark. We conclude this section by mentioning a few facts that did not appear in Theorems 1 and 1' or in their proofs, but may be of relevance for other developments related to these theorems.

1^o Let t be a number in $(0, 1)$, and let μ, μ', μ'' be distributions in $\mathcal{D}_{\text{alg}}(k)$ such that:

$$(i) R_{\mu'} = \frac{1}{1-t} R_{\mu}, \quad \text{and} \quad (ii) \eta_{\mu''} = t \eta_{\mu'}.$$

Then a direct computation using the relations between R -transforms and η -series yields

$$R_{\mu''} = \frac{t}{1-t} \eta_{\mu}. \quad (5.7)$$

Let us observe moreover that if $\mu \in \mathcal{D}_c(k)$ then μ' and μ'' belong to $\mathcal{D}_c(k)$ as well; this is because $\mathcal{R}_c(k)$ is closed under multiplication by $1/(1-t)$, and $\mathcal{E}_c(k)$ is closed under multiplication by t . In the case when $t = 1/2$, these observations can be used to give an alternative proof for the inclusion (5.5) in the proof of Lemma 5.7.

2^o A positivity phenomenon which was observed in preceding work on multi-variable \boxplus -infinite divisibility is the following. Let f be a series in $\mathcal{R}_c^{\text{inf-div}}(k)$ and let $\mu \in \mathcal{D}_{\text{alg}}(k)$ be the distribution determined by the formula

$$\mu(X_{i_1} \cdots X_{i_n}) = \text{Cf}_{(i_1, \dots, i_n)}(f), \quad \forall n \geq 1, \forall 1 \leq i_1, \dots, i_n \leq k.$$

Then $\mu(P^*P) \geq 0$ for every polynomial $P \in \mathbb{C}\langle X_1, \dots, X_k \rangle$ which has no constant term. Moreover, this positivity can be used to construct realizations of \boxplus -infinitely divisible distributions by operators on the full Fock space. For more details on this, see Sections 4.5 and 4.7 of [11].

3^o A property of distributions $\mu \in \mathcal{D}_c(k)$ which is often considered is *traciality* ($\mu(PQ) = \mu(QP)$ for every $P, Q \in \mathbb{C}\langle X_1, \dots, X_k \rangle$). This did not appear in Theorems 1 and 1', and in fact the multi-variable Boolean Bercovici-Pata bijection does not preserve traciality. From a combinatorial perspective, the cause of this fact is that the natural action of the cyclic group \mathbb{Z}_n on partitions of $\{1, \dots, n\}$ does not leave invariant the set $\text{Int}(n)$ of interval partitions.

6. A special property of the partial order \ll

The goal of this section is to prove a combinatorial result which lies at the heart of the proofs of Theorems 2 and 2', and which will be stated precisely in Corollary 6.11. The proof

of this result will use the various facts about non-crossing partitions that were reviewed in Section 2, tailored to the special situation of parity-preserving partitions $\theta \in NC(2n)$.

6.1 Remark. Let n be a positive integer. We refer to the above Section 2.2 for the definition of what it means that a partition $\theta \in NC(2n)$ is parity-preserving, and for the fact that such a partition can always be uniquely presented in the form $\theta = \pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ where $\pi, \rho \in NC(n)$ are such that $\rho \leq K(\pi)$.

A useful remark is that for $\theta = \pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ as above we always have

$$|\theta| = |\pi| + |\rho| \geq |\pi| + |K(\pi)| = n + 1, \quad (6.1)$$

with the equality $|\theta| = n + 1$ holding if and only if $\rho = K(\pi)$.

If $\theta \in NC(2n)$ is parity-preserving, then the blocks A of θ such that $A \subseteq \{1, 3, \dots, 2n-1\}$ will be called **odd blocks**, while the blocks B of θ such that $B \subseteq \{2, 4, \dots, 2n\}$ will be called **even blocks**.

Let us observe that a parity-preserving partition $\theta \in NC(2n)$ always has at least two outer blocks. Indeed, the odd block M such that $M \ni 1$ and the even block N such that $N \ni 2n$ are distinct, and both have to be outer. If these M and N are the only outer blocks of θ , then we will say (naturally) that θ **has exactly two outer blocks**. In view of Remark 2.9.2, a necessary and sufficient condition for this to happen is that

$$\min(N) = \max(M) + 1. \quad (6.2)$$

If θ is written as $\pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ with $\pi, \rho \in NC(n)$ such that $\rho \leq K(\pi)$, then it is immediately seen that the condition (6.2) amounts to

$$\min(N_0) = \max(M_0), \quad (6.3)$$

where N_0 is the block of ρ such that $N_0 \ni n$ and M_0 is the block of π such that $M_0 \ni 1$. The condition (6.3) is nicely expressed in terms of the permutations P_π, P_ρ associated to π and ρ in Remark 2.1. Indeed, it is immediate that $\min(N_0) = P_\rho(n)$ and $\max(M_0) = P_\pi^{-1}(1)$, so in the end we arrive to the following equivalence: for $\pi, \rho \in NC(n)$ such that $\rho \leq K(\pi)$ we have that

$$\left(\begin{array}{c} \pi^{(\text{odd})} \sqcup \rho^{(\text{even})} \text{ has exactly} \\ \text{two outer blocks} \end{array} \right) \Leftrightarrow P_\rho(n) = P_\pi^{-1}(1). \quad (6.4)$$

6.2 Remark. Let θ be a parity-preserving partition in $NC(2n)$, and let us consider the Hasse diagram for the embracing partial order on the set of blocks of θ . This is exactly as in Remark 2.11 of Section 2, with the additional ingredient that the vertices of the Hasse diagram are now bicoloured (indeed, a vertex of the Hasse diagram is a block of θ , and is of one of the colours “even” or “odd”). We warn the reader that, in general, this bicolouring does not turn the Hasse diagram into a so-called “bipartite graph” (i.e. it is not precluded that an edge of the Hasse diagram connects two vertices of the same colour); this issue is discussed in more detail in Remark 6.5 and in Lemmas 6.6 and 6.8 below.

In what follows, instead of talking explicitly about edges of the Hasse diagram, we will prefer to use the related concept of “parenthood” for the vertices of the diagram. This concept is defined in the general framework of a rooted forest, and goes as follows. Let A be a vertex of a rooted forest, and suppose that A is not a root of the forest. Let B be the unique root which lies in the same connected component of the forest as A , and consider

the unique path (A_0, A_1, \dots, A_s) from A to B in the forest (with $s \geq 1$, $A_0 = A$, $A_s = B$). The vertex A_1 of this path is then called the *parent* of the vertex A .

In the case of the specific rooted forest that we are dealing with here (Hasse diagram for embracing partial order on blocks), the concept of parenthood can of course be also given directly in terms of embracings of blocks. We leave it as an immediate exercise to the reader to check that, when proceeding on this line, the definition comes out as follows.

6.3 Definition. Let θ be a parity-preserving partition in $NC(2n)$, and let A be a block of θ , such that A is not outer. Then there exists a block P of θ , uniquely determined, with the following two properties:

- (i) $P \subseteq A$, $P \neq A$, and (ii) If $A' \subseteq A$ and $A' \neq A$ then $A' \subseteq P$.

This block P will be called the **parent of A with respect to θ** , and will be denoted as $\text{Parent}_\theta(A)$.

The next proposition states a few basic properties of the parenthood relation defined above. The verifications of these properties are immediate, and are left as exercise to the reader.

6.4 Proposition. Let θ be a parity-preserving partition in $NC(2n)$.

1° Let A be a block of θ such that $\max(A) < 2n$. Let B be the block of θ such that $\max(A) + 1 \in B$, and suppose that $\max(A) + 1$ is not the minimal element of B . Then A is not outer, and $\text{Parent}_\theta(A) = B$.

2° Let A, B be blocks of θ such that $\min(B) = \max(A) + 1$. Then either both A and B are outer blocks, or none of them is, and in the latter case we have that $\text{Parent}_\theta(A) = \text{Parent}_\theta(B)$.

3° Let A be a block of θ which is not outer, denote $\text{Parent}_\theta(A) =: B$, and suppose that A and B have the same parity. Let θ' be the partition of $\{1, \dots, 2n\}$ which is obtained from θ by joining together the blocks A and B . Then θ' is a parity-preserving partition in $NC(2n)$, and we have $\theta \ll \theta'$.

6.5 Remark. Let θ be a parity-preserving partition in $NC(2n)$, and consider the Hasse diagram of the embracing partial order on blocks of θ . We will next look at the question of when is it possible that an edge of the Hasse diagram connects two blocks of the same parity. It is clear that if two blocks of θ are connected by an edge of the Hasse diagram, then there is one of the two blocks which is the parent of the other; due to this fact, the question stated above amounts to asking when is it possible that the blocks A and $\text{Parent}_\theta(A)$ have the same parity (where A is a non-outer block of θ). This will be addressed in the Lemmas 6.6 and 6.8 below. We will actually be only interested in the situation when θ has exactly two outer blocks.

6.6 Lemma. Let π be a partition in $NC(n)$, and consider the parity-preserving partition $\theta := \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})} \in NC(2n)$.

1° θ has exactly two outer blocks.

2° If A is a block of θ which is not outer, then the block $\text{Parent}_\theta(A)$ has parity opposite from the parity of A .

Proof. 1° In view of the above equivalence (6.4), it suffices to observe that $P_{K(\pi)}(n) = P_{\pi}^{-1}(P_{1_n}(n)) = P_{\pi}^{-1}(1)$ (where we used the formula for $P_{K(\pi)}$ given in Equation (2.3) of Remark 2.3).

2° We will present the argument in the case when A is an odd block of θ . (The case when A is an even block is treated similarly, and is left as exercise.) We have $A = \{2a-1 \mid a \in A_o\}$ where A_o is a block of π . Let us denote $\min(A_o) = a'$ and $\max(A_o) = a''$. We have $a' > 1$ (from $a' = 1$ it would follow that $A \ni 1$, and A would be an outer block of θ). Observe that

$$P_{K(\pi)}(a' - 1) = P_{\pi}^{-1}(P_{1_n}(a' - 1)) = P_{\pi}^{-1}(a') = a''.$$

This shows that $a' - 1$ and a'' belong to the same block B_o of $K(\pi)$. The block $B = \{2b \mid b \in B_o\}$ of θ will then contain the elements $2(a' - 1) = \min(A) - 1$ and $2a'' = \max(A) + 1$. From Proposition 6.4.1 we infer that $\text{Parent}_{\theta}(A) = B$, and in particular it follows that $\text{Parent}_{\theta}(A)$ has parity opposite from the one of A , as required. **QED**

6.7 Remark. Let us look again at the discussion from Remark 6.2, but where now θ is taken to be of the special kind from the preceding lemma, $\theta := \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$ for a partition $\pi \in NC(n)$. So we look at the Hasse diagram for the embracing partial order on blocks of θ , and we note that in this case the Hasse diagram *is* bipartite – indeed, Lemma 6.6 assures us that every edge of the Hasse diagram connects an even block of θ with an odd block of θ .

The discussion around the Hasse diagram for this special θ can be pushed a bit further, in order to establish a connection with the theory of graphs on surfaces. We make here a brief comment on how this goes. (The comment will not be used in what follows, but may be illuminating for a reader who is familiar with the theory of graphs on surfaces, and prefers to translate various facts about $NC(n)$ into the language of that theory.) So let us note another feature of our Hasse diagram: it has exactly two connected components, rooted at the two outer blocks of θ , and where one of the roots (the outer block containing 1) is odd while the other (the outer block containing $2n$) is even. Let us add to the Hasse diagram an edge which connects the two roots. The graph we obtain is a tree with $n + 1$ vertices (and n edges), where the vertices are bicoloured, and where every edge connects two vertices of different colours; moreover, this tree has a “marked” edge, namely the edge that was added to the Hasse diagram. This kind of structure (a bicoloured bipartite tree with one marked edge) plays an important role in the theory of graphs on surfaces, where among other things it is used to produce a special factorization “of genus zero” of the cycle $1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$; see for instance section 1.5 of the monograph [6]. In particular, the bicoloured tree we just encountered (by starting from $\theta := \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$, and by adding an edge to the corresponding Hasse diagram) can be used to retrieve the factorization “ $P_{\pi}P_{K(\pi)} = P_{1_n}$ ” reviewed in the Remark 2.3 – hence this bicoloured tree completely determines the partition π we started with.

The next lemma is in some sense a converse of Lemma 6.6.

6.8 Lemma. Let π and ρ be partitions in $NC(n)$ such that $\rho \leq K(\pi)$, and consider the parity-preserving partition $\theta := \pi^{(\text{odd})} \sqcup \rho^{(\text{even})} \in NC(2n)$. Consider the following properties that θ may have:

- (i) θ has exactly two outer blocks.
- (ii) If A is a block of θ which is not outer, then the block $\text{Parent}_{\theta}(A)$ has parity opposite from the parity of A .

If both (i) and (ii) hold, then $\rho = K(\pi)$.

Proof. Let M and N be the blocks of θ such that $M \ni 1$ and $N \ni 2n$. Hypothesis (i) implies that $\min(N) = 1 + \max(M)$ (cf. Equation (6.2) in Remark 6.1). We denote

$$m := \min(N)/2 = (1 + \max(M))/2 \in \{1, \dots, n\}.$$

Let \mathfrak{N} be the set of blocks of θ which are not outer, and define a function $F : \mathfrak{N} \rightarrow \{1, \dots, n\}$ by the formula:

$$F(X) := \begin{cases} (1 + \max(X))/2 & \text{if } X \text{ is an odd block} \\ \min(X)/2 & \text{if } X \text{ is an even block.} \end{cases}$$

It is immediate that $F(X) \neq m$, $\forall X \in \mathfrak{N}$, hence that $\text{card}(F(\mathfrak{N})) \leq n - 1$.

We claim that $F(X) \neq F(Y)$ whenever $X, Y \in \mathfrak{N}$ are blocks of opposite parities. Indeed, assume for instance that X is odd, Y is even, and $F(X) = F(Y) =: i$. Then $\min(Y) = 2i = 1 + \max(X)$, and Proposition 6.4.2 implies that $\text{Parent}_\theta(X) = \text{Parent}_\theta(Y)$. But this is not possible, since (by hypothesis (ii)) the block $\text{Parent}_\theta(X)$ is even, while $\text{Parent}_\theta(Y)$ is odd.

It then follows that the function F is one-to-one. Indeed, if $X, Y \in \mathfrak{N}$ are such that $F(X) = F(Y)$ then X and Y must have the same parity (by the claim proved in the preceding paragraph), and it immediately follows that $X \cap Y \neq \emptyset$, hence that $X = Y$.

From the injectivity of F we conclude that $\text{card}(\mathfrak{N}) = \text{card}(F(\mathfrak{N})) \leq n - 1$, and that the total number of blocks of θ is $|\theta| = 2 + \text{card}(\mathfrak{N}) \leq n + 1$. But it was noticed in Remark 6.1 that $\theta = \pi^{(\text{odd})} \sqcup \rho^{(\text{even})}$ has $|\theta| \geq n + 1$, with equality if and only if $\rho = K(\pi)$. The conclusion of the lemma immediately follows. **QED**

6.9 Lemma. Let π be in $NC(n)$, and let θ be a parity-preserving partition in $NC(2n)$ such that $\theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$. Let X, Y be blocks of θ such that $Y = \text{Parent}_\theta(X)$, and suppose that Y has the same parity as X . Let θ' be the partition of $\{1, \dots, 2n\}$ which is obtained from θ by joining together the blocks X and Y . Then θ' also is a parity-preserving partition in $NC(2n)$ such that $\theta' \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$.

Proof. We will write the argument in the case when X and Y are even blocks of θ . The case when X and Y are odd blocks is similar, and is left as exercise.

The fact that θ' is a parity-preserving partition in $NC(2n)$ follows from Proposition 6.4.3, the point of the proof is to verify that $\theta' \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$.

Let us write $\theta = \sigma^{(\text{odd})} \sqcup \tau^{(\text{even})}$ with $\sigma, \tau \in NC(n)$. Since we assumed that X, Y are even blocks of θ , we thus have $X = \{2a \mid a \in A\}$ and $Y = \{2b \mid b \in B\}$ with A, B blocks of τ . It is clear that $\theta' = \sigma^{(\text{odd})} \sqcup (\tau')^{(\text{even})}$, where $\tau' \in NC(n)$ is obtained from τ by joining together its blocks A and B .

Consider the partition $\phi := \pi^{(\text{odd})} \sqcup \tau^{(\text{even})}$ of $\{1, \dots, 2n\}$; that is, ϕ is obtained by putting together the even blocks of θ with the odd blocks of $\pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$. Note that $\phi \in NC(2n)$; this is because we have

$$\sigma^{(\text{odd})} \sqcup \tau^{(\text{even})} = \theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}, \quad (6.5)$$

which implies in particular that $\tau \leq K(\pi)$. It is clear that X and Y are blocks of ϕ . We observe the following facts.

Fact 1. Y is the block-parent of X with respect to ϕ .

Verification of Fact 1. Assume by contradiction that there exists a block Z of ϕ such that $Z \neq X, Y$ and $Y \subseteq Z \subseteq X$. The block Z cannot be even – in that case it would be a block of θ , and the embracings $Y \subseteq Z \subseteq X$ would contradict the hypothesis that $\text{Parent}_\theta(X) = Y$. So Z is odd, and consequently it is a block of $\pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$. Since $\theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$, there exists a block W of θ which has $\min(W) = \min(Z)$ and $\max(W) = \max(Z)$. But then the embracings $Y \subseteq Z \subseteq X$ are equivalent to $Y \subseteq W \subseteq X$; since $W \neq X, Y$ (which happens because W is odd, while X, Y are even), we again obtain a contradiction with the hypothesis that $\text{Parent}_\theta(X) = Y$. (End of verification of Fact 1.)

Fact 2. $\tau' \leq K(\pi)$.

Verification of Fact 2. Let ϕ' be the partition obtained from ϕ by joining together its blocks X and Y . It is clear that $\phi' = \pi^{(\text{odd})} \sqcup (\tau')^{(\text{even})}$; thus proving the inequality $\tau' \leq K(\pi)$ is equivalent to proving that $\phi' \in NC(2n)$ (see the equivalence (2.1) in Remark 2.2). But ϕ' is indeed in $NC(2n)$, as we see by invoking the above Fact 1 and Proposition 6.4.3. (End of verification of Fact 2.)

We conclude the argument as follows: look at the inequalities $\sigma \ll \pi$ and $\tau \ll K(\pi)$ which are implied by (6.5), and combine the second of these inequalities with Fact 2, in order to get that $\tau' \ll K(\pi)$. (Indeed, from $\tau \leq \tau' \leq K(\pi)$ and $\tau \ll K(\pi)$ we get that $\tau' \ll K(\pi)$ – this follows directly from how \ll is defined.) So we have $\sigma \ll \pi$ and $\tau' \ll K(\pi)$, which entails that $\theta' = \sigma^{(\text{odd})} \sqcup (\tau')^{(\text{even})} \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$, as required. **QED**

6.10 Proposition. Let θ be a parity-preserving partition in $NC(2n)$, which has exactly two outer blocks. There exists a unique partition $\pi \in NC(n)$ such that $\theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$.

Proof. Let us denote

$$\mathcal{T} := \{\theta' \in NC(2n) \mid \theta' \text{ parity-preserving and } \theta' \gg \theta\}.$$

Observe that every $\theta' \in \mathcal{T}$ has exactly two outer blocks (because θ is like that, and by the equivalence (2.10) in Remark 2.12).

Let $\tilde{\theta} \in \mathcal{T}$ be an element which is maximal with respect to the partial order \ll ; that is, $\tilde{\theta}$ is such that if $\theta' \in \mathcal{T}$ and $\theta' \gg \tilde{\theta}$, then $\theta' = \tilde{\theta}$.

Let X be a block of $\tilde{\theta}$ which is not outer, and let us denote $Y := \text{Parent}_{\tilde{\theta}}(X)$. We claim that X and Y have opposite parities. Indeed, in the opposite case the partition θ' obtained from $\tilde{\theta}$ by joining the blocks X and Y together would still be in \mathcal{T} , and would satisfy $\theta' \gg \tilde{\theta}$, $\theta' \neq \tilde{\theta}$, contradicting the maximality of $\tilde{\theta}$.

We thus see that $\tilde{\theta}$ satisfies the hypotheses of Lemma 6.8, and must therefore be of the form $\pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$ for some $\pi \in NC(n)$. This proves the existence part of the lemma.

For the uniqueness part, suppose that $\pi, \rho \in NC(n)$ are such that $\theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$ and $\theta \ll \rho^{(\text{odd})} \sqcup K(\rho)^{(\text{even})}$. We then consider the set

$$\mathcal{S} := \left\{ \theta' \in NC(2n) \mid \begin{array}{l} \theta' \text{ parity-preserving, } \theta' \gg \theta, \\ \theta' \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})} \text{ and } \theta' \ll \rho^{(\text{odd})} \sqcup K(\rho)^{(\text{even})} \end{array} \right\},$$

and we let $\hat{\theta}$ be a maximal element in (\mathcal{S}, \ll) .

Let X be a block of $\hat{\theta}$ which is not outer, and let us denote $Y := \text{Parent}_{\hat{\theta}}(X)$. We claim that X and Y have opposite parities. Indeed, in the opposite case the partition θ' obtained

from $\widehat{\theta}$ by joining the blocks X and Y together would still be in \mathcal{S} – where the inequalities $\theta' \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$ and $\theta' \ll \rho^{(\text{odd})} \sqcup K(\rho)^{(\text{even})}$ follow from Lemma 6.9. This partition θ' would satisfy $\theta' \gg \widehat{\theta}$ and $\theta' \neq \widehat{\theta}$, contradicting the maximality of $\widehat{\theta}$.

We thus see that $\widehat{\theta}$ satisfies the hypotheses of Lemma 6.8, and must therefore be of the form $\sigma^{(\text{odd})} \sqcup K(\sigma)^{(\text{even})}$ for some $\sigma \in NC(n)$. But then the inequality $\widehat{\theta} \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$ implies that $\sigma \leq \pi$ and $K(\sigma) \leq K(\pi)$, which in turn imply that $\sigma = \pi$ (as $K(\sigma) \leq K(\pi)$ is equivalent to $\sigma \geq \pi$). A similar argument shows that $\sigma = \rho$, and the desired equality $\pi = \rho$ follows. **QED**

6.11 Corollary. Let σ, τ be two partitions in $NC(n)$. The following two statements are equivalent:

(1) There exists $\pi \in NC(n)$ such that $\sigma \ll \pi$ and $\tau \ll K(\pi)$.

(2) $\tau \leq K(\sigma)$ and the associated permutations P_σ and P_τ are such that $P_\sigma^{-1}(1) = P_\tau(n)$. Moreover, in the case when the statements (1) and (2) are true, the partition π with the properties stated in (1) is uniquely determined.

Proof. “(1) \Rightarrow (2)” Fix $\pi \in NC(n)$ such that $\sigma \ll \pi$ and $\tau \ll K(\pi)$. We have in particular that $\sigma \leq \pi$ and $\tau \leq K(\pi)$. So $\tau \leq K(\pi) \leq K(\sigma)$, and the inequality $\tau \leq K(\sigma)$ follows. On the other hand it is immediate that we have the implications

$$\sigma \ll \pi \Rightarrow P_\sigma^{-1}(1) = P_\pi^{-1}(1) \quad \text{and} \quad \tau \ll K(\pi) \Rightarrow P_\tau(n) = P_{K(\pi)}(n).$$

So the relation $P_{K(\pi)}(n) = P_\pi^{-1}(1)$ observed in the proof of Lemma 6.6.1 implies that $P_\tau(n) = P_\sigma^{-1}(1)$.

“(2) \Rightarrow (1)” Consider the partition $\theta = \sigma^{(\text{odd})} \sqcup \tau^{(\text{even})}$ of $\{1, \dots, 2n\}$. The hypotheses in (2) give us that $\theta \in NC(2n)$ and that it has exactly two outer blocks (where we also use the equivalence (6.4) from Remark 6.1). Hence, by Proposition 6.10, there exists $\pi \in NC(n)$ such that $\theta \ll \pi^{(\text{odd})} \sqcup K(\pi)^{(\text{even})}$, and the latter inequality is clearly equivalent to having that $\sigma \ll \pi$ and $\tau \ll K(\pi)$. Observe moreover that here we also obtain the uniqueness of π : if $\rho \in NC(n)$ is such that $\sigma \ll \rho$ and $\tau \ll K(\rho)$, then it follows that $\theta \ll \rho^{(\text{odd})} \sqcup K(\rho)^{(\text{even})}$, and the uniqueness part in Proposition 6.10 implies that $\rho = \pi$. **QED**

7. Proofs of Theorems 2 and 2'

7.1 Remark. The calculations made in this section will use the explicit formula for the coefficients of a boxed convolution $f \boxtimes g$, where f and g are series in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. This formula says that for every $n \geq 1$ and every $1 \leq i_1, \dots, i_n \leq k$ we have

$$\text{Cf}_{(i_1, \dots, i_n)}(f \boxtimes g) = \sum_{\pi \in NC(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(f) \cdot \text{Cf}_{(i_1, \dots, i_n); K(\pi)}(g). \quad (7.1)$$

One can moreover extend Equation (7.1) to a formula which describes the generalized coefficients $\text{Cf}_{(i_1, \dots, i_n); \pi}$ (as introduced in Notation 3.2.3) for the series $f \boxtimes g$. This says that for every $n \geq 1$, every $1 \leq i_1, \dots, i_n \leq k$, and every $\rho \in NC(n)$ we have

$$\text{Cf}_{(i_1, \dots, i_n); \rho}(f \boxtimes g) = \sum_{\substack{\pi \in NC(n), \\ \pi \leq \rho}} \text{Cf}_{(i_1, \dots, i_n); \pi}(f) \cdot \text{Cf}_{(i_1, \dots, i_n); K(\pi)}(g). \quad (7.2)$$

For a discussion of how one arrives to these formulas, we refer to [8], Lecture 17.

Instead of proving directly the Theorems 2 and 2', we will prove yet another equivalent reformulation of these theorems, which is stated as follows.

7.2 Theorem. Let f, g be two series in $\mathbb{C}_0\langle\langle z_1, \dots, z_k \rangle\rangle$. Then

$$\text{Reta}(f \boxtimes g) = \text{Reta}(f) \boxtimes \text{Reta}(g). \quad (7.3)$$

Proof. We fix $n \geq 1$ and $1 \leq i_1, \dots, i_n \leq k$, and we will prove the equality of the coefficients of $z_{i_1} \cdots z_{i_n}$ of the series appearing on the two sides of (7.3). On the left-hand side we have:

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(\text{Reta}(f \boxtimes g)) &= \sum_{\substack{\rho \in NC(n), \\ \rho \ll 1_n}} \text{Cf}_{(i_1, \dots, i_n); \rho}(f \boxtimes g) \quad (\text{by Proposition 3.9.1}) \\ &= \sum_{\substack{\rho \in NC(n), \\ \rho \ll 1_n}} \left(\sum_{\substack{\sigma \in NC(n), \\ \sigma \leq \rho}} \text{Cf}_{(i_1, \dots, i_n); \sigma}(f) \cdot \text{Cf}_{(i_1, \dots, i_n); K_\rho(\sigma)}(g) \right) \quad (\text{by Eqn. (7.2)}) \\ &= \sum_{\sigma, \tau \in NC(n)} N'(\sigma, \tau) \text{Cf}_{(i_1, \dots, i_n); \sigma}(f) \cdot \text{Cf}_{(i_1, \dots, i_n); \tau}(g), \end{aligned}$$

where for $\sigma, \tau \in NC(n)$ we denoted

$$N'(\sigma, \tau) := \text{card}\{\rho \in NC(n) \mid \rho \ll 1_n, \sigma \leq \rho, K_\rho(\sigma) = \tau\}. \quad (7.4)$$

On the right-hand side of Equation (7.3) the corresponding coefficient is:

$$\begin{aligned} \text{Cf}_{(i_1, \dots, i_n)}(\text{Reta}(f) \boxtimes \text{Reta}(g)) &= \sum_{\pi \in NC(n)} \text{Cf}_{(i_1, \dots, i_n); \pi}(\text{Reta}(f)) \cdot \text{Cf}_{(i_1, \dots, i_n); K(\pi)}(\text{Reta}(g)) \\ &= \sum_{\pi \in NC(n)} \left(\sum_{\substack{\sigma \in NC(n), \\ \sigma \ll \pi}} \text{Cf}_{(i_1, \dots, i_n); \sigma}(f) \right) \cdot \left(\sum_{\substack{\tau \in NC(n), \\ \tau \ll K(\pi)}} \text{Cf}_{(i_1, \dots, i_n); \tau}(g) \right) \quad (\text{by Eqn. (3.21)}) \\ &= \sum_{\sigma, \tau \in NC(n)} N''(\sigma, \tau) \text{Cf}_{(i_1, \dots, i_n); \sigma}(f) \cdot \text{Cf}_{(i_1, \dots, i_n); \tau}(g), \end{aligned}$$

where for $\sigma, \tau \in NC(n)$ we denoted

$$N''(\sigma, \tau) := \text{card}\{\pi \in NC(n) \mid \sigma \ll \pi, \tau \ll K(\pi)\}. \quad (7.5)$$

From the above calculations it is clear that (7.3) will follow if we can prove that

$$N'(\sigma, \tau) = N''(\sigma, \tau), \quad \forall \sigma, \tau \in NC(n).$$

Now, the content of Corollary 6.11 is that

$$N''(\sigma, \tau) = \begin{cases} 1, & \text{if } \tau \leq K(\sigma) \text{ and } P_\tau(n) = P_\sigma^{-1}(1) \\ 0, & \text{otherwise.} \end{cases} \quad (7.6)$$

So it remains to prove that $N'(\sigma, \tau)$ is also described by the right-hand side of (7.6).

Let us observe that always $N'(\sigma, \tau) \in \{0, 1\}$. Indeed, if there exists $\rho \in NC(n)$ such that $\rho \geq \sigma$ and $K_\rho(\sigma) = \tau$, then ρ is uniquely determined – this is because the associated permutation P_ρ is determined, $P_\rho = P_\sigma P_\tau$. In order to prove that $N'(\sigma, \tau)$ is equal to the quantity on the right-hand side of (7.6) it will therefore suffice to verify the following equivalence:

$$\left(\tau \leq K(\sigma) \text{ and } P_\tau(n) = P_\sigma^{-1}(1) \right) \Leftrightarrow \left(\begin{array}{c} \exists \rho \in NC(n) \text{ such that} \\ \rho \ll 1_n, \sigma \leq \rho, \text{ and } K_\rho(\sigma) = \tau \end{array} \right). \quad (7.7)$$

Verification of “ \Rightarrow ” in (7.7). Consider the relative Kreweras complement of τ in $K(\sigma)$, and then consider the partition

$$\rho := K^{-1}(K_{K(\sigma)}(\tau)) \in NC(n).$$

From Equation (2.5) in Remark 2.3 we have that $K_{K(\sigma)}(\tau) \leq K(\tau)$; if we then apply the order-reversing map K^{-1} to both sides of this inequality, we get that $\rho \geq \sigma$. An immediate calculation involving the permutations associated to ρ, σ and τ gives us that $P_\rho = P_\sigma P_\tau$, and this in turn implies that $K_\rho(\sigma) = \tau$. Finally, observe that $P_\rho(n) = P_\sigma P_\tau(n) = 1$ (with the latter inequality following from the fact that $P_\sigma^{-1}(1) = P_\tau(n)$). This shows that $\rho \ll 1_n$, and completes this verification.

Verification of “ \Leftarrow ” in (7.7). Let $\rho \in NC(n)$ be such that $\rho \ll 1_n$, $\rho \geq \sigma$, and $K_\rho(\sigma) = \tau$. From Equation (2.5) in Remark 2.3 we obtain that $\tau = K_\rho(\sigma) \leq K(\sigma)$. On the other hand the permutations associated to ρ, σ, τ satisfy $P_\tau = P_\sigma^{-1} P_\rho$ (because $\tau = K_\rho(\sigma)$), and $P_\rho(n) = 1$ (because $\rho \ll 1_n$). Hence we have $P_\tau(n) = P_\sigma^{-1}(P_\rho(n)) = P_\sigma^{-1}(1)$, as required. **QED**

7.3 Remark (*proofs of Theorems 2 and 2'*). In the introduction section it was shown how Theorem 2 is derived from Theorem 2', and here we show how Theorem 2' follows from the above Theorem 7.2. Let μ and ν be two distributions from $\mathcal{D}_{\text{alg}}(k)$. Consider the formula (1.13) which is satisfied by μ and ν , and apply Reta to both its sides. We obtain

$$\begin{aligned} \text{Reta}(R_{\mu \boxtimes \nu}) &= \text{Reta}(R_\mu \boxtimes R_\nu) \\ &= \text{Reta}(R_\mu) \boxtimes \text{Reta}(R_\nu) \quad (\text{by Theorem 7.2}). \end{aligned}$$

Since Reta maps an R -transform to the η -series of the same distribution, we have thus obtained that $\eta_{\mu \boxtimes \nu} = \eta_\mu \boxtimes \eta_\nu$, as stated in Theorem 2'.

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